# Application of Continuous-Discrete Conversion and Balanced Truncation Algorithm to the Order Reduction Problem of Unstable Systems

**Ngo Kien Trung**
Hanoi Industrial Textile Garment University, Hanoi, Vietnam
trungnk@hict.edu.vn

**Vu Thi Anh Ngoc**
Technical and Economic College, Thai Nguyen, Vietnam
v.t.a.ngoc@gmail.com

**Nguyen Thi Tham**
Technical and Economic College, Thai Nguyen, Vietnam
nguyenthamcdktkt@gmail.com

**Vu Ngoc Kien**
Research Development Institute of Advanced Industrial Technology (RIAT), Thai Nguyen University of Technology, Thai Nguyen, Vietnam
kienvn@tnut.edu.vn (corresponding author)

*Received: 12 November 2024 | Revised: 7 December 2024, 19 December 2024, and 28 December 2024 | Accepted: 1 January 2025*

 *DOI: https://doi.org/10.48084/etasr.9586*

## ABSTRACT

**Since the model order reduction problem was first posed, numerous order reduction algorithms have been proposed in a variety of approaches. However, the majority of these algorithms have been developed to reduce the order of stable systems. In certain practical applications, such as high-order controller design, the original system may be unstable. Consequently, there is a need for order reduction algorithms capable of reducing the order of both stable and unstable systems. The present paper focuses on introducing a Continuous-Discrete (CD) transformation-based Balanced Truncation (BT) algorithm, which has the capacity to reduce the order of both stable and unstable systems. The efficiency of the improved BT algorithm is demonstrated by the simulation results.**

***Keywords-model order reduction; high-order controller; balanced truncation algorithm; stable system; unstable system***

## I. INTRODUCTION

In the mathematical description of dynamic systems, there is a preference for models that are detailed and accurate. However, the pursuit of such fidelity often results in intricate and sophisticated mathematical models of high complexity and order. This, in turn, poses significant challenges in the domains of simulation and control design. Consequently, there is a demand for methodologies that facilitate the simplification of complex, high-order mathematical models, thereby yielding a simple, low-order algorithm that ensures an almost accurate description of the dynamic system. This necessity has given rise to a plethora of algorithms, which have collectively formed the field of MOR. Among the MOR algorithms, the BT algorithm [1] has gained particular popularity. This algorithm is founded on the simultaneous diagonalization of both the control gramian and the observation gramian of the original system. The matrix that facilitates the diagonalization of both the control and observation gramians of the original system results in the conversion of the original system, represented in any basis system, to an equivalent balanced basis system, which is referred to as the internal balance space system. The elimination of states that contribute little to the input-output relationship of the original system, or states that are less controllable and observable, results in the attainment of a reduced-order system with fewer states or lower order. The order reduction error of Moore's BT algorithm [1] is minimal, constituting the primary advantage of this algorithm. Authors in [2, 3] normalized the BT algorithm and the relationship with the Hankel norm was determined. Authors in [4] conducted a thorough analysis of the BT algorithm, demonstrating its efficacy through illustrative examples. Authors in [5] introduced and analyzed the BT algorithm and the appropriate time algorithm and illustrative examples were provided. Through this analysis, it was determined that the BT algorithm provides good global order reduction error, while the appropriate time algorithm provides good system behavior error results. In addition, authors in [6, 7] proposed and demonstrated the correctness of using the BT algorithm for the semi-discrete Stroke equation, while authors in [8] developed a formula to determine the order reduction error of the BT algorithm and concurrently established the concept of frequency-weighted BT order reduction. Authors in [9] proposed an extended BT algorithm by constructing the concept of extended gramians. These gramians are determined by solving a set of linear matrix inequalities. This approach is particularly advantageous in the context of frequency-weighted reduction. Authors in [10-12] made adjustments to the BT algorithms to meet the order reduction requirements of each specific problem. However, it should be noted that the BT algorithms proposed in [1-12] are only capable of working with stable linear systems. However, numerous mathematical models of dynamic systems or controllers [13-16] are in fact unstable linear systems. Consequently, MOR algorithms must possess the capacity to function with both stable and unstable linear systems.

The application of the BT algorithm to unstable linear systems is a subject that has been extensively researched [15-26]. Authors in [24] performed the transformation of an unstable continuous system into a stable continuous system by shifting the coordinate origin. This renders the system eligible for the application of the BT algorithm. The subsequent application of the BT algorithm to reduce the order of the stable system results in the attainment of a stable reduced order system. Finally, the reverse projection is performed to convert the reduced order system from a stable form to an unstable form. A parallel can be drawn between authors in [25] and in [24], as both involved a transformation of a continuous system into a stable one, followed by a reduction in order to make it eligible for the BT algorithm implementation. Nevertheless, the approach presented in [25] involves transforming an unstable discrete system into a stable discrete system, hence ensuring its eligibility for the BT algorithm application. Building upon the findings [24, 25] and the CD transformation, this study proposes a BT algorithm based on the CD transformation, with illustrative examples demonstrating the algorithm's efficacy. A distinguishing feature of the BT algorithm is its capacity for a flexible application of the CD transformation, enabling its usage with unstable continuous linear systems, a capability not possessed by Moore's BT algorithm [1]. Additionally, a comparative analysis will be conducted to assess the efficacy of the proposed algorithm in reducing the order of unstable linear systems.

## II. CONTINUOUS - DISCRETE CONVERSION

### A. γ-Stable Discrete Time System

Consider a discrete linear system:

$$\boldsymbol{x}(\boldsymbol{k}+\mathbf{1}) = \boldsymbol{A_d x}(\boldsymbol{k}) + \boldsymbol{B_d u}(\boldsymbol{k}), \boldsymbol{y}(\boldsymbol{k}) = \boldsymbol{C_d x} + \boldsymbol{D_d u}(\boldsymbol{k}) \qquad (1)$$

where $(\boldsymbol{A}_d, \boldsymbol{B}_d, \boldsymbol{C}_d, \boldsymbol{D}_d) \in R^{nxn} \times R^{\mathrm{nxm}} \times R^{\mathrm{kxn}} \times R^{\mathrm{kxm}}$, $x(k) \in R^n, u(k) \in R^m, y(k) \in R^k$, $x(k)$, $x(k+1)$ represent the state sequence or state trajectory, $u(k)$ is the input sequence, $y(k)$ the output sequence, $A_d$ is the system matrix, and $B_d$, $C_d$ and $D_d$ are, respectively, the input matrix, the output matrix, and the transmission matrix. The transfer function of the system (1) is:

$$G(z) := C_d(zI - A_d)^{-1}B_d + D_d, z \in C$$

- Definition 1: Discrete-time system (1) is called $\alpha$-stable if the real part of the poles $|\boldsymbol{\lambda}(\boldsymbol{A_d})| < \alpha, \alpha \geq \mathbf{1}$, where $\boldsymbol{D_\alpha}$ is the set of $\alpha$- stable discrete-time systems.

With the above definition, the discrete system is asymptotically stable when the condition $\boldsymbol{\gamma} = \mathbf{1}$ is satisfied, at this point we can call the stable discrete system – 0. The matrix $\boldsymbol{A_d}$ of the asymptotically stable discrete system will have the form of a Schur matrix $|\boldsymbol{\lambda}(\boldsymbol{A_d})| < 1$.

### B. λ-Stable Continuous System

Let us consider the continuous linear system:

$$\dot{x}(t) = \boldsymbol{A}_c x(t) + \boldsymbol{B}_c \boldsymbol{u}(t), y(t) = \boldsymbol{C}_c x(t) + \boldsymbol{D}_c \boldsymbol{u}(t) \qquad (2)$$

where $(\boldsymbol{A}_c, \boldsymbol{B}_c, \boldsymbol{C}_c, \boldsymbol{D}_c) \in R^{nxn} \times R^{\mathrm{nxm}} \times R^{\mathrm{kxn}} \times R^{\mathrm{kxm}}$, $x(t) \in R^n, u(t) \in R^m, y(t) \in R^k$, $x(t)$ is the state vector, $u(t)$ the input excitation vector, $y(t)$ the output measurement vector, $\boldsymbol{A_c}$ is the system matrix, $\boldsymbol{B_c}$, $\boldsymbol{C_c}$, and $\boldsymbol{D_c}$ are, respectively, the input matrix, the output matrix, and the transmission matrix. The transfer function of the system (2) is:

$$\boldsymbol{G}(s) := \boldsymbol{C}_c(s\boldsymbol{I} - \boldsymbol{A}_c)^{-1}\boldsymbol{B}_c + \boldsymbol{D}_c, \mathrm{s} \in C$$

- Definition 2. The continuous system (2) is called $\lambda$ -stable if the real part of the poles real $(\lambda(\mathbf{A})) < \lambda, \lambda \geq 0$. $C_\lambda$ is the set of $\lambda$ - stable continuous systems.

With the above definition, the continuous system (2) is called asymptotically stable continuous system when it satisfies the control $\lambda = 0$and the matrix $\boldsymbol{A}$ of the system will be the Huzwitz matrix with real $(\lambda(\mathbf{A})) < 0$.

### C. Continuous-Discrete Conversion

- Definition 3. The following conversion: $\Omega_{\lambda,\alpha}: C_\lambda \to D_\alpha$ $(\mathbf{A}_c, \mathbf{B}_c, \mathbf{C}_c, \mathbf{D}_c) \to (\mathbf{A}_d, \mathbf{B}_d, \mathbf{C}_d, \mathbf{D}_d)$ where $\mathbf{A}_d = \alpha(\mathbf{I} - \bar{\mathbf{A}}_c)^{-1}(\mathbf{I} + \bar{\mathbf{A}}_c)$, $\mathbf{B}_d = \sqrt{2\alpha}(\mathbf{I} - \bar{\mathbf{A}}_c)^{-1}\mathbf{B}_c$, $\mathbf{C}_d = \sqrt{2\alpha}\mathbf{C}_c(\mathbf{I} - \bar{\mathbf{A}}_c)^{-1}$, $\mathbf{D}_d = \mathbf{D}_c + \mathbf{C}_c(\mathbf{I} - \bar{\mathbf{A}}_c)^{-1}\mathbf{B}_c, \bar{\mathbf{A}}_c = \mathbf{A}_c - \lambda\mathbf{I}$ is called CD conversion. This conversion transforms a $\lambda$-stable continuous system to an $\alpha$-stable discrete system. The CD conversion performs two transformations at the same time, namely converting the system from a continuous system to a discrete system and converting the system from an unstable system to a stable system. There is also a reverse conversion: $\Omega^{-1}_{\alpha,\lambda}: D_\alpha \to C_\lambda$ $(\boldsymbol{A}_d, \boldsymbol{B}_d, \boldsymbol{C}_d, \boldsymbol{D}_d) \to (\boldsymbol{A}_c, \boldsymbol{B}_c, \boldsymbol{C}_c, \boldsymbol{D}_c)$, where $\boldsymbol{A}_c = \lambda\boldsymbol{I} + (\boldsymbol{I} + \bar{\boldsymbol{A}}_d)^{-1}(\bar{\boldsymbol{A}}_d - \boldsymbol{I})$, $\boldsymbol{B}_c = \sqrt{\frac{2}{\alpha}}(\boldsymbol{I} + \bar{\boldsymbol{A}}_d)^{-1}\boldsymbol{B}_d$, $\boldsymbol{C}_c =$

$\sqrt{\frac{2}{\alpha}}\boldsymbol{C}_d(\boldsymbol{I}+\bar{\boldsymbol{A}}_d)^{-1}$ , $\boldsymbol{D}_c=\boldsymbol{D}_d-\frac{1}{\alpha}\boldsymbol{C}_d(\boldsymbol{I}+\bar{\boldsymbol{A}}_d)^{-1}, \bar{\boldsymbol{A}}_d=\frac{A}{\alpha}$ is called reverse CD conversion. This conversion transforms an $\alpha$-stable discrete system to a $\lambda$-stable continuous system. The reverse CD conversion performs two transformations at the same time, namely converting the system from a discrete system to a continuous system and converting the system from a stable system to an unstable system.

## III. BALANCED TRUNCATION ALGORITHM BASED ON THE CONTINUOUS-DISCRETE CONVERSION

### A. MOR Problem

Consider a linear system described by:

$$\dot{x}(t)=\mathbf{A}x(t)+\mathbf{Bu}(t), y(t)=\boldsymbol{C}x(t)+\boldsymbol{Du}(t) \quad (3)$$

where $(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})\in R^{nxn}\times R^{\mathrm{nxm}}\times R^{\mathrm{kxn}}\times R^{\mathrm{kxm}}$ , $x(t)\in R^n, u(t)\in R^m, y(t)\in R^k$, $x(t)$ is the state vector, $u(t)$ the input excitation vector, $y(t)$ the output measurement vector, $\boldsymbol{A}$ is the system matrix $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ are, respectively, the input matrix, the output matrix, and the transmission matrix. The objective of the MOR problem is to determine a linear system of order $r$ described by:

$$\dot{x}_r(t)=\mathbf{A}_r x(t)+\mathbf{B}_r\mathbf{u}(t), y_r(t) = \mathbf{C}_r x(t)+\mathbf{D}_r\mathbf{u}(t) \quad (4)$$

where $(\mathbf{A}_r,\mathbf{B}_r,\mathbf{C}_r,\mathbf{D}_r)\in R^{\mathrm{rxr}}\times R^{\mathrm{rxm}}\times R^{\mathrm{kxr}}\times R^{\mathrm{kxm}}$ , $x(t)\in R^r, u(t)\in R^m,\ y(t)\in R^k$, with $r < n$, so that (4) can replace (3) in control system design, simulation, or analysis problems.

### B. Balanced Truncation Algorithm based on the Continuous-Discrete Conversion

As previously mentioned, when (3) is in an unstable linear form, this system will not satisfy the condition for solving the Lyapunov equation. Consequently, it is impossible to determine the control gramian and the observed gramian. This, in turn, results in the BT algorithm [1] not being applicable to the unstable linear system. To solve the MOR problem by the BT algorithm when system (3) is in an unstable continuous linear form, the BT algorithm is proposed based on the CD conversion [2] as follows:

- Input: $\lambda$-stable continuous system $\mathbf{G}(s):=\mathbf{C}_c(s\mathbf{I}-\mathbf{A}_c)^{-1}\mathbf{B}_c+\mathbf{D}_c\in C_\lambda$.
- Step 1: convert the system from a λ-stable continuous system into the stable discrete system $-\ \mathbf{0}\ \widehat{\mathbf{G}}_d(z)\in R$ by conversion $(\widehat{\mathbf{A}}_d,\widehat{\mathbf{B}}_d,\widehat{\mathbf{C}}_d,\widehat{\mathbf{D}}_d)=\Omega_{\lambda,1}(\mathbf{A}_c,\mathbf{B}_c,\mathbf{C}_c,\mathbf{D}_c)$.
- Step 2: Apply the BT algorithm for the system $(\widehat{\mathbf{A}}_d,\widehat{\mathbf{B}}_d,\widehat{\mathbf{C}}_d,\widehat{\mathbf{D}}_d)$ , yielding a discrete reduced-order system$\widehat{\mathbf{G}}_{rd}(z)$.
- Step 3: Convert the discrete reduced-order system $\widehat{\mathbf{G}}_{rd}(z)$ into λ-stable reduced-order system $\widehat{\mathbf{G}}_{rc}(s)$ by mapping $(\widehat{\mathbf{A}}_{rc},\widehat{\mathbf{B}}_{rc},\widehat{\mathbf{C}}_{rc},\widehat{\mathbf{D}}_{rc})=\Omega_{\lambda,1}^{-1}(\widehat{\mathbf{A}}_{rd},\widehat{\mathbf{B}}_{rd},\widehat{\mathbf{C}}_{rd},\widehat{\mathbf{D}}_{rd})$.
- Output: reduced-order continuous system $\widehat{\mathbf{G}}_{rc}(s)=\widehat{\mathbf{C}}_{rc}(s\mathbf{I}-\widehat{\mathbf{A}}_{rc})\widehat{\mathbf{B}}_{rc}+\widehat{\mathbf{D}}_{rc}$.

By applying the CD conversion to stable continuous systems or unstable continuous systems in step 1, the BT algorithm in step 2 will only work with stable discrete systems. The reduced order system obtained after step 2 will be a stable discrete system. Applying the inverse CD conversion in step 3 will help conver the reduced order system back to the same form as the original system. Thus, by flexibly applying the CD conversion and inverse CD conversion, the BT algorithm can reduce the order for both the stable continuous systems and unstable continuous systems and preserve the properties (stable or unstable) of the original system in the reduced order system.

## IV. APPLICATION OF THE BALANCED TRUNCATION ALGORITHM BASED ON THE CONTINUOUS-DISCRETE CONVERSION

In order to provide a clear demonstration of the effectiveness of the algorithm in reducing the order of unstable linear systems, the present study will apply this algorithm to an unstable linear system as described in [15]. The 5th-order unstable model [15] is expressed as:

$$\boldsymbol{T}(s)=\frac{10^3.s2-2.1209.10^{-8}s+0.11925}{s^5+0.3s^4+0.54s^3+0.192s^2-0.064s}$$

The objective of this study is to reduce the order of an unstable system of order 5 by identifying a reduced order system that can substitute for the original system while maintaining the properties of the original system as closely as possible. The order reduction of the unstable system $\boldsymbol{T}(s)$ is achieved through the implementation of the BT algorithm, which is based on the CD transformation. This process is:

Step 1: The poles of the system $\boldsymbol{T}(s)$ are 0, 0.8i, -0.8i, -0.5, and 0.2, and we choose $\lambda$ = 1.4. The system $\boldsymbol{T}(s)$ when converted to a stable discrete form has the form:

$$\boldsymbol{T}(z)=\frac{58.78\,z^5+195.9\,z^4+236.8\,z^3+122.5\,z^2+24.5z+1.634}{z^5+1.068\,z^4+0.504\,z^3+0.1232\,z^2+0.01424z+0.0005878}$$

The poles of the $\boldsymbol{T}(z)$ system: -0.1667, -0.25+ 0.25i, -0.25 -0.25i, -0.0909 -0.3103.

Step 2: Result of the order reduction $\boldsymbol{T}(z)$ according to the BT algorithm, as shown in Table I.

TABLE I. RESULT OF THE ORDER REDUCTION OF THE SYSTEM $\boldsymbol{T}$(Z)

| Order | Reduced-order system |
|---|---|
| 4 | $\frac{58.78z^4+186.1z^3+205.7z^2+88.17z+9.797}{z^4+0.9012z^3+0.3538z^2+0.06425z+0.003524}$ |
| 3 | $\frac{58.78\,z^3+175.9z^2+175.3z+58.19}{z^3+0.727z^2+0.2313z+0.02425}$ |
| 2 | $\frac{58.78z^2+141.8z+90.01}{z^2+0.1476z+0.0933}$ |

Step 3: Convert the reduced order discrete system to an unstable reduced order linear system, the result is as displayed in Table II. MATLAB software is used to represent and compare the step response and bandwidth response of the reduced-order systems and original systems. In Figures 1 and 2, a comparative analysis of the step responses for the original system and the reduced-order systems is presented.

TABLE II. RESULT OF THE ORDER REDUCTION OF THE UNSTABLE SYSTEM $\boldsymbol{T}$(S)

| Order | Reduced-order system |
|---|---|
| 4 | $\frac{-6.016.10^{-7}s^4 + 1.546.10^{-5}s^3 - 0.0001774s^2 + 1000s + 0.01}{s^4 + 0.3s^3 + 0.5399s^2 + 0.192s - 0.06409}$ |
| 3 | $\frac{0.03963s^3 - 0.7895s^2 + 6.803s + 967.2}{s^3 + 0.2043s^2 + 0.6779s + 0.03619}$ |
| 2 | $\frac{0.03963s^3 - 0.7895s^2 + 6.803s + 967.2}{s^3 + 0.2043s^2 + 0.6779s + 0.03619}$ |

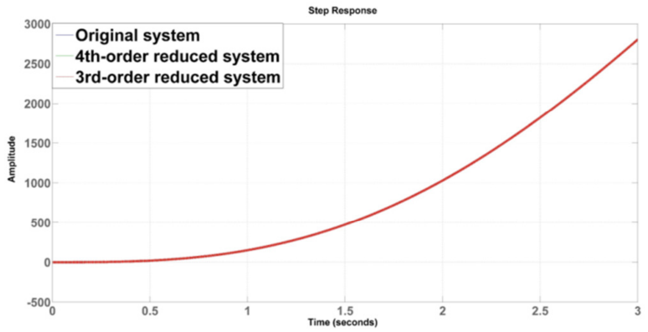


Fig. 1. Step response of the original system and the reduced-order systems.

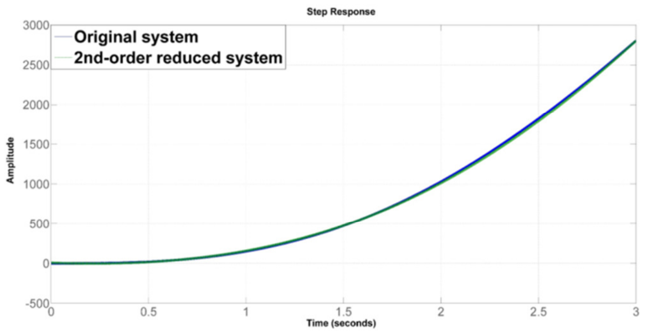


Fig. 2. Step response of the original system and the 2nd- order reduced system.

As presented in Figure 1, the step response of the fourth-order and 3rd-order reduced systems is in complete alignment with the step response of the original system. Figure 2 reveals a slight discrepancy between the step response of the 2nd-order reduced system and that of the original system. Figure 3 presents the phase response of both the original and reduced-order systems.

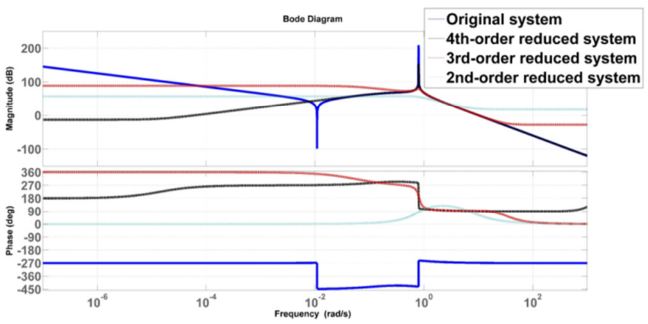


Fig. 3. Bode response of original and reduced order systems.

In the frequency region w > 0.0209 rad/s, the Frequency Amplitude Response (FAR) of the 4th-order reduced system is in complete alignment with the FAR of the original system. Conversely, in the frequency region w < 0.0209 rad/s, the FAR of the 4th-order reduced system exhibits a complete deviation from the FAR of the original system. In the frequency range of 0.869 rad/s < w < 22.2 rad/s, the FAR of the 3rd-order reduced system exhibits a complete coincidence with that of the original system. Conversely, in the frequency range of w < 0.869 rad/s and w > 22.2 rad/s, the FAR of the 3rd-order reduced system demonstrates a complete deviation from that of the original system. A complete divergence is observed between the FAR of the 2nd-order reduced system and the FAR of the original system. Furthermore, the Frequency Phase Response (FPR) of the reduced order systems is distinct from the FPR of the original system. The 2nd-order reduced system has the capacity to substitute for the original system, provided that the discrepancy in the frequency amplitude response between the original system and the reduced-order system is negligible. The 3rd-order reduced system can substitute for the original system if the discrepancy in the frequency amplitude response between the original system and the reduced-order system is minimal, and if the difference in the frequency phase response between the original system and the reduced system is disregarded. Authors in [15] achieved an order reduction of the original system $\boldsymbol{T}(s)$ through the implementation of the BT algorithm based on projection, analogous to the one proposed in [24] BT. The resultant system is a 3rd-order reduced system:

$$\frac{-22.112s^2+132.64s+655.6}{s^3-0.19596s^2+0.78147s-0.079718}$$

The step response of the 2nd-order reduction system and the 3rd-order reduced system are presented in [15]. The step response of the 2nd-order reduced system according to the BT algorithm based on the CD conversion exhibits a strong resemblance to the step response of the original system. In contrast, the step response of the 3rd-order reduced system, deviates significantly from the step response of the original system. In comparison with the results reported in [15], the order reduction system in this study exhibits a reduced order reduction error and a lower order. The combination of the CD conversion with the BT algorithm has been demonstrated to expand the application of the algorithm and improve the order reduction results. Authors in [27] proposed a method combining the Krylov subspace algorithm with the genetic algorithm. The decomposition of the order reduction process into two steps, with step 1 using the Krylov subspace algorithm and step 2 employing the genetic algorithm, facilitates the determination of the optimal nominal order reduction model. The approach outlined in [27] is notable for its computational efficiency and analytical simplicity.

## V. CONCLUSIONS

The paper presents a thorough exposition of the Balanced Truncation (BT) algorithm, founded upon the principle of Continuous-Discrete (CD) conversion. The BT algorithm, when implemented with the CD conversion, facilitates the order reduction for unstable systems. The efficacy of the BT algorithm based on the CD conversion is demonstrated by its application to the problem of simplifying a 5th-order unstable system, which results in the replacement of the original system with a 2nd-order reduced system. The efficacy of the BT algorithm based on the CD conversion is substantiated by the simulation results. However, it was also observed that when this algorithm is applied to the problem of simplifying a 5th-order unstable system, the order reduction results obtained are superior to those attained with the BT algorithm [15]. In subsequent studies, the influence of the transformation coefficient value of the algorithm on the model order reduction results will be evaluated, and methods to combine the BT algorithm with optimization algorithms, such as genetic algorithms, will be investigated. It is important to note that this algorithm's efficacy is constrained to the reduction of order for linear systems; its inability to reduce the order for nonlinear systems is a significant limitation. This limitation will be addressed in subsequent studies.

## ACKNOWLEDGMENT

The authors would like to thank Thai Nguyen University of Technology, Vietnam for their facilities and constructive criticism of the manuscript.

## REFERENCES

[1] B. Moore, "Principal component analysis in linear systems: Controllability, observability, and model reduction," *IEEE Transactions on Automatic Control*, vol. 26, no. 1, pp. 17–32, Feb. 1981, https://doi.org/10.1109/TAC.1981.1102568.

[2] L. Pernebo and L. Silverman, "Model reduction via balanced state space representations," *IEEE Transactions on Automatic Control*, vol. 27, no. 2, pp. 382–387, Apr. 1982, https://doi.org/10.1109/TAC.1982.1102945.

[3] K. Glover, "All optimal Hankel-norm approximations of linear multivariable systems and their L, ∞ -error bounds," *International Journal of Control*, vol. 39, no. 6, pp. 1115–1193, Jun. 1984, https://doi.org/10.1080/00207178408933239.

[4] A. C. Antoulas, "Approximation of Large-Scale Dynamical Systems: An Overview," *IFAC Proceedings Volumes*, vol. 37, no. 11, pp. 19–28, Jul. 2004, https://doi.org/10.1016/S1474-6670(17)31584-7.

[5] A. C. Antoulas, D. C. Sorensen, and S. Gugercin, "A survey of model reduction methods for large-scale systems," in *Structured Matrices in Mathematics, Computer Science, and Engineering I*, American Mathematical Society, 2001, pp. 193–219.

[6] T. Stykel, "Gramian-Based Model Reduction for Descriptor Systems," *Mathematics of Control, Signals and Systems*, vol. 16, no. 4, pp. 297–319, Mar. 2004, https://doi.org/10.1007/s00498-004-0141-4.

[7] D. F. Enns, "Model reduction with balanced realizations: An error bound and a frequency weighted generalization," in *The 23rd IEEE Conference on Decision and Control*, Las Vegas, NV, USA, Dec. 1984, pp. 127–132, https://doi.org/10.1109/CDC.1984.272286.

[8] H. Sandberg, "Model reduction of linear systems using extended balanced truncation," in *2008 American Control Conference*, Seattle, WA, USA, Jun. 2008, pp. 4654–4659, https://doi.org/10.1109/ACC.2008.4587229.

[9] M. Heinkenschloss, D. C. Sorensen, and K. Sun, "Balanced Truncation Model Reduction for a Class of Descriptor Systems with Application to the Oseen Equations," *SIAM Journal on Scientific Computing*, vol. 30, no. 2, pp. 1038–1063, Jan. 2008, https://doi.org/10.1137/070681910.

[10] T. Stykel, "Balanced truncation model reduction for semidiscretized Stokes equation," *Linear Algebra and its Applications*, vol. 415, no. 2, pp. 262–289, Jun. 2006, https://doi.org/10.1016/j.laa.2004.01.015.

[11] I. P. Duff, P. Goyal, and P. Benner, "Balanced Truncation for a Special Class of Bilinear Descriptor Systems," *IEEE Control Systems Letters*, vol. 3, no. 3, pp. 535–540, Jul. 2019, https://doi.org/10.1109/LCSYS.2019.2911904.

[12] H. H. Bui, "The Application of LQG Balanced Truncation Algorithm to the Digital Filter Design Problem," *Engineering, Technology & Applied Science Research*, vol. 12, no. 6, pp. 9458–9463, Dec. 2022, https://doi.org/10.48084/etasr.5235.

[13] K. Mustaqim, D. K. Arif, E. Apriliani, and D. Adzkiya, "Model reduction of unstable systems using balanced truncation method and its application to shallow water equations," *Journal of Physics: Conference Series*, vol. 855, no. 1, Jun. 2017, Art. no. 012029, https://doi.org/10.1088/1742-6596/855/1/012029.

[14] D. Novella Rodríguez, B. Del Muro Cuéllar, O. Sename, and M. Velasco Villa, "On the stabilization of high order systems with two unstable poles plus time delay," in *2012 20th Mediterranean Conference on Control & Automation (MED)*, Barcelona, Spain, Jul. 2012, pp. 12–17, https://doi.org/10.1109/MED.2012.6265607.

[15] J. Yang, C. S. Chen, J. a. D. Abreu-Garcia, and Y. Xu, "Model reduction of unstable systems," *International Journal of Systems Science*, vol. 24, no. 12, pp. 2407–2414, Dec. 1993, https://doi.org/10.1080/00207729308949638.

[16] H. B. Minh, C. B. Minh, and V. Sreeram, "Balanced Generalized Singular Perturbation Method for Unstable Linear Time Invariant Continuous Systems," *Acta Mathematica Vietnamica*, vol. 42, no. 4, pp. 615–635, Dec. 2017, https://doi.org/10.1007/s40306-017-0215-2.

[17] N. Mirnateghi and E. Mirnateghi, "Model reduction of unstable systems using balanced truncation," in *2013 IEEE 3rd International Conference on System Engineering and Technology*, Shah Alam, Malaysia, Aug. 2013, pp. 193–196, https://doi.org/10.1109/ICSEngT.2013.6650169.

[18] C. S. Hsu and D. Hou, "Reducing unstable linear control systems via real schur transformation," *Electronics Letters*, vol. 27, no. 11, pp. 984–986, May 1991, https://doi.org/10.1049/el:19910614.

[19] S. K. Nagar and S. K. Singh, "An algorithmic approach for system decomposition and balanced realized model reduction," *Journal of the Franklin Institute*, vol. 341, no. 7, pp. 615–630, Nov. 2004, https://doi.org/10.1016/j.jfranklin.2004.07.005.

[20] A. Varga, "Model reduction software in the SLICOT library," in *CACSD. Conference Proceedings. IEEE International Symposium on Computer-Aided Control System Design (Cat. No.00TH8537)*, Anchorage, AK, USA, Sep. 2000, pp. 249–254, https://doi.org/10.1109/CACSD.2000.900219.

[21] Fatmawati, R. Saragih, R. T. Bambang, and Y. Soeharyadi, "Balanced truncation for unstable infinite dimensional systems via reciprocal transformation," *International Journal of Control, Automation and Systems*, vol. 9, no. 2, pp. 249–257, Apr. 2011, https://doi.org/10.1007/s12555-011-0206-3.

[22] E. Jonckheere and L. Silverman, "A new set of invariants for linear systems–Application to reduced order compensator design," *IEEE Transactions on Automatic Control*, vol. 28, no. 10, pp. 953–964, Oct. 1983, https://doi.org/10.1109/TAC.1983.1103159.

[23] K. Zhou, G. Salomon, and E. Wu, "Balanced realization and model reduction for unstable systems," *International Journal of Robust and Nonlinear Control*, vol. 9, no. 3, pp. 183–198, 1999, https://doi.org/10.1002/(SICI)1099-1239(199903)9:3<183::AID-RNC399>3.0.CO;2-E.

[24] A. Zilouchian, "Balanced structures and model reduction of unstable systems," in *IEEE Proceedings of the SOUTHEASTCON '91*, Williamsburg, VA, USA, Apr. 1991, pp. 1198–1201 vol.2, https://doi.org/10.1109/SECON.1991.147956.

[25] C. Boess, N. Nichols, and A. Bunse-Gerstner, "Model reduction for discrete unstable control systems using a balanced truncation approach." University of Reading, 2010.

[26] C. Boess, A. S. Lawless, N. K. Nichols, and A. Bunse-Gerstner, "State estimation using model order reduction for unstable systems," *Computers & Fluids*, vol. 46, no. 1, pp. 155–160, Jul. 2011, https://doi.org/10.1016/j.compfluid.2010.11.033.

[27] A. Tamri, A. B. H. Adamou-Mitiche, and L. Mitiche, "Model order reduction, a novel method using krylov sub-spaces and genetic algorithm," *Studies in Engineering and Exact Sciences*, vol. 5, no. 1, pp. 525–543, Mar. 2024, https://doi.org/10.54021/seesv5n1-030.