Radial Displacements in a Rotating Disc of Uniform Thickness Made of Functionally Graded Material

Vasile Nastasescu
Military Technical Academy "Ferdinand I", Romania
nastasescuv@gmail.com (corresponding author)

Antonela Toma
National University of Science and Technology, Politehnica Bucharest, Romania
antonela2222@yahoo.com

Received: 7 December 2023 | Revised: 24 December 2023 | Accepted: 7 January 2024

 DOI: https://doi.org/10.48084/etasr.6713

ABSTRACT

The finite element method is used to calculate a rotating disc, which has a uniform thickness and is made of functionally graded materials, based on the concepts of multilayer disc and equivalent material. These concepts are also available for analytical calculus. The multilayered disc concept perceives the disc as constructed from several layers, and the equivalent material concept regards the disc material as composed of homogeneous and isotropic material but with fictitious properties equivalent in behavior to the functionally graded material. These two concepts, encompassed in this study, allow us to contemplate the variation according to the material law and Poisson's ratio, which is often neglected, to reduce the mathematical complexity. The concepts, models, and methods involved in this study were validated by employing numerical and analytical calculations. The proposed method introduced simplicity, precision, and accessibility to solve the complex problem of functionally graded structures. The calculus development, model validation, and result analysis were based on numerical calculus using the finite element method. The utilized models were grounded on the existence of an axial-symmetric plane. So, 2D or 3D simplified models can be used with several variants regarding the mesh fineness. This study results and models are useful to specialists and structure designers of this type, have a high degree of generality, and present opportunities for the application of other calculation methods.

Keywords-functionally graded material; equivalent material model; multi-layer wall; rotating disc; FEM

I. INTRODUCTION

The appearance and continuous development of Functionally Graded Materials (FGMs) have become a subject of scientific research based on at least two important aspects: their manufacture and their calculation. FGMs are very important to engineers, designers, and manufacturers because of their specific properties. They are considered composite materials [1-2], but they are considered as new materials [3]. This study investigates the problem of calculating structures from FGMs, focusing on the calculus of a rotating disc having a uniform thickness made of FGM. This study aimed to develop methods and models as accessible and efficient as possible to successfully solve calculus problems of structures made of FGMs. Analytical calculation of structures from FGMs raises mathematical difficulties in general [1, 4], which arise from the fact that engineers no longer work with material constants but with functions that describe the material properties.

Many material property laws exist. The problem becomes more complicated when considering this aspect for all properties [5-6]. For example, the Poisson ratio variation is often neglected, but the difficulties do not completely disappear [5-7]. This study introduces two concepts for approaching the calculation of FGM structures with an acceptable accuracy [8-10]: the multilayer structure concept and the concept of equivalent material (structure). This paper displays the calculus results of a rotating disc having a uniform thickness and made of FGMs using these two concepts. In addition, a comparative quantitative study on the influence of considering the Poisson ratio variation is exhibited [11]. The novelty of this study comprises a new approach development, based on the two original concepts, providing a method and practical calculation models for a widely used structure made of FGM. The proposed approach allows for viewing the Poisson ratio variation according to the adopted material law. To simplify the mathematical calculation, the variation of the Poisson ratio in the calculation of functionally graded structures was neglected.

II. FUNCTIONALLY GRADED DISC

FGMs represent a category of materials that can be called composites but whose elastic and physical properties along one of their dimensions, namely Young's modulus, shear modulus, density, Poisson ratio, thermal or electrical conductivity, etc., are described by material adopted laws (models) [1, 4, 8]. Usually, one of the materials is a high-resistance material (ceramic) and the other is a common material having low resistance, like aluminum, steel, etc. Such materials are increasingly being used in engineering design. Plates, beams, and tubes are perhaps the most commonly utilized structural elements of FGMs. Figure 1 shows a rotating disc that has a uniform thickness and is made of FGMs, where the highest material properties are on its outer contour and the lowest material properties are on the inner contour where the working pressure is also applied. Figure 2 presents a quarter of the rotating disc portrayed in Figure 1, built in the variation version on the thickness wall of the material properties.

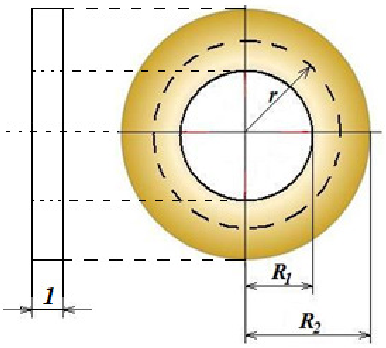


Fig. 1. Thick-walled tube under internal pressure.

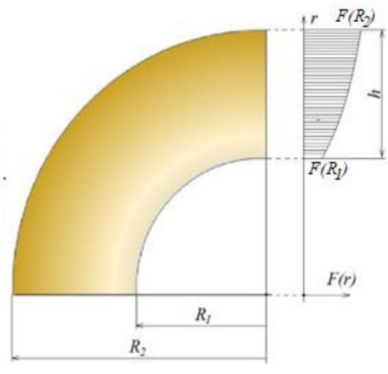


Fig. 2. Material property variation in FGMs.

The highest values are toward the outside (radius R_2) as the function $F(r)$ varies. There are many material property laws. The most used one is the power law [1, 4, 6, 8], which can be written as:

$$F(r) = F_b + (F_t - F_b)\left(\frac{r-R_1}{h}\right)^k \quad (1)$$

$$R_1 < r < R_2 \quad (2)$$

where $F(r)$ refers to any material property such as Young's modulus $E(r)$, Poisson ratio $\nu(r)$, density $\rho(r)$, etc., F_b is the $F(r)$ value for $r = R_1$, F_t is the $F(r)$ value for $r = R_2$, h is the disc thickness, r is the coordinate that describes the position of a point in the wall material, h is the wall thickness and k is a coefficient (power coefficient), which may have different values, less than or greater than 1. The indices b and t refer to the extreme edges of the material: b for the lower face ($r=R_1$) and t for the upper face ($r=R_2$).

The power coefficient k influence is very important [8], as can be seen in Figures 3 and 4. The qualitative evolution of the relative values of Young's modulus $E(r)$ and Poisson's ratio $\nu(r)$ versus the value of the relative radius and the power coefficient k can be followed by examining Figures 3 and 4. The evolution curves of the density are similar to the curves for $E(r)$. The calculation of structures from FGMs raises a series of difficulties, caused by the variation of material properties, even if the variation occurs according to a known law [12-14]. To overcome certain challenges, this study introduced two calculus concepts disclosed in the following case study.

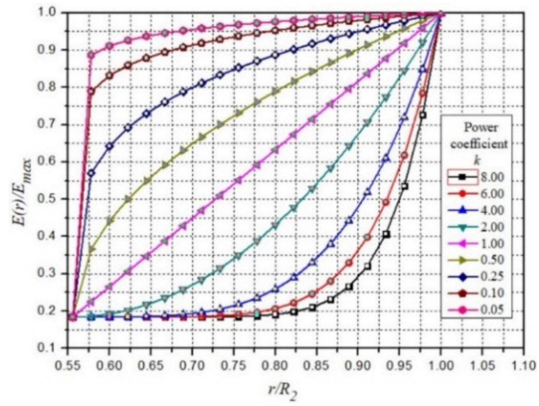


Fig. 3. Relative Young's modulus versus r and k.

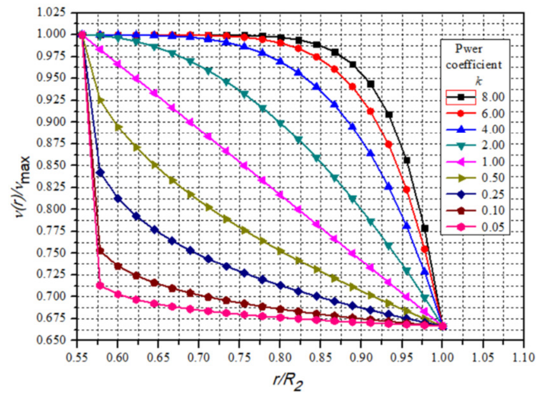


Fig. 4. Relative Poisson's ratio versus r and k.

III. CASE STUDY

This study examines a rotating disc of uniform thickness, made of FGM with an inner radius $R_1 = 0.005$ m and $R_2 = 0.005$ m, without any applied pressure, but with a rotation of 628 rad/s (n = 6000 rot/min). The disc was constructed from an FGM based on two materials having the properties showcased in Table I.

TABLE I. MATERIAL PROPERTIES

Materials:	Ceramic	Aluminum
Position	**Top**	**Bottom**
E [Pa]	3.8E+11	7.0E+10
V [-]	0.22	0.33
P [kg/m^3]	3960	2700

The material law is the power law represented by (1). The solution of the case study consists of the calculus of the disc wall radial displacements. The objective is achieved using three models with finite elements in five discretization variants to highlight the influence of the number of layers of the considered model. This study also accentuates the impact of contemplating the variation of Poisson's ratio or adopting different constant values.

IV. THE MULTILAYER WALL CONCEPT

The calculus of the rotating disc, having a uniform thickness, and made of FGMs, using the concept of multilayer wall, regards the disc wall reconstructed from overlapping layers of material, each layer having constant material properties (as a homogeneous and isotropic material), as shown in Figure 5. The property values of each layer are calculated based on the material law of FGM, for an average value of the coordinate corresponding to the middle of each layer [9-10]. Therefore, the continuous variation of the material properties is replaced by a variation in steps, defined by the respective layers. A greater number of considered layers leads to a closer solution to reality [10].

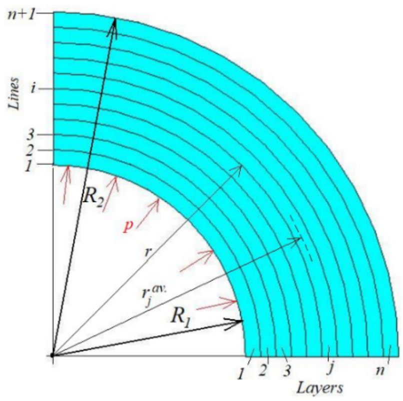


Fig. 5. Multilayer tube wall.

Choosing the layer number is an important issue to which this study gives useful answers. Using the multilayer wall concept for the thick-walled tube, 5, 10, 20, 30, and 40 layers were successively adopted. For each layer, a constant value of the material characteristics was calculated, applying the material power law according to (1), where the coordinate r is the average value of the r coordinates that describe the separation lines of the respective layer ($r_j^{av} \equiv r_{layer}^{av}$, as shown in Figure 6 and Table II). Table II displays the values of Young's modulus and Poisson's ratio for the case of 20 layers and k = 0.05.

TABLE II. YOUNG'S MODULUS, POISSON RATIO, AND DENSITY VALUES FOR EACH LAYER, FOR $k = 0.05$

Layer No.	Average radius r_{layer}^{av}	Young's modulus $E(r)$	Poisson ratio $\nu(r)$	Density $\rho(r)$
-	[m]	[Pa]	[-]	[kg/m³]
1	0.0051	3.278E+11	0.239	3747.8
2	0.0053	3.423 E+11	0.233	3806.9
3	0.0055	3.494 E+11	0.231	3835.6
4	0.0057	3.541 E+11	0.229	3854.8
5	0.0059	3.577 E+11	0.228	3869.4
6	0.0061	3.606 E+11	0.227	3881.2
7	0.0063	3.631 E+11	0.226	3891.1
8	0.0065	3.652 E+11	0.225	3899.7
9	0.0067	3.670 E+11	0.225	3907.2
10	0.0069	3.687 E+11	0.224	3914.0
11	0.0071	3.702 E+11	0.224	3920.1
12	0.0073	3.715 E+11	0.223	3925.6
13	0.0075	3.728 E+11	0.223	3930.7
14	0.0077	3.740 E+11	0.222	3935.5
15	0.0079	3.751 E+11	0.222	3939.9
16	0.0081	3.761 E+11	0.222	3944.0
17	0.0083	3.770 E+11	0.221	3947.9
18	0.0085	3.779 E+11	0.221	3951.6
19	0.0087	3.788 E+11	0.221	3955.1
20	0.0089	3.796 E+11	0.221	3958.4
Average values:		3.654 E+11	0.225	3900.8

The curves in Figures 6 and 7, constructed with the values in Table II, comparatively illustrate the multilayer disc wall concept. The evolution in steps in the rotating disc thickness is compared to the continuous evolution. For both properties, in the inner surface proximity, the steps are large and then they decrease, being close to the continuous curve. This aspect is explained by the very different properties of these two materials and the very low power coefficient. Its increase leads to flattening of the curve and even the change of the curvature center. The large variation between steps moves to the other extreme surface, with the increase in the power coefficient value. It can be observed that the power coefficient k influence is very important.

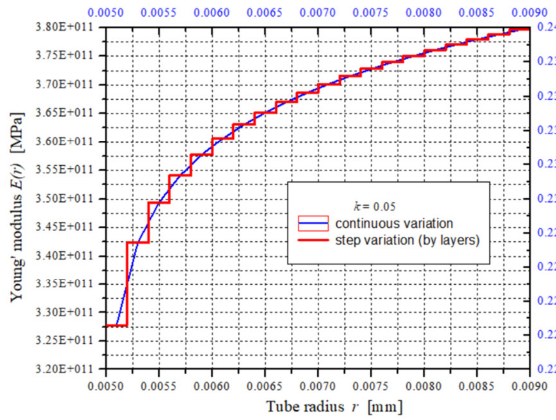


Fig. 6. Young's modulus step variation.

V. THE EQUIVALENT MATERIAL CONCEPT

The equivalent material concept applies to the given structure without any modification of its geometry and dimensions and consists of replacing the FGM with a homogeneous and isotropic material with equivalent properties. These properties are determined as the average of the calculated values, based on the material law and a number of lines or layers, as shown in Figure 6. Using the concept of equivalent material for a disc with uniform thickness means replacing the real properties with fictitious ones for a homogeneous and isotropic material with the same geometry. It was found that the equivalent value (equivalent material concept) of the entire FGM was the arithmetic mean of the values considered for each layer or line. Table III presents the equivalent Young's modulus for different values of the power coefficient as well as 5, 10, 20, 30, 40, and 80 layers.

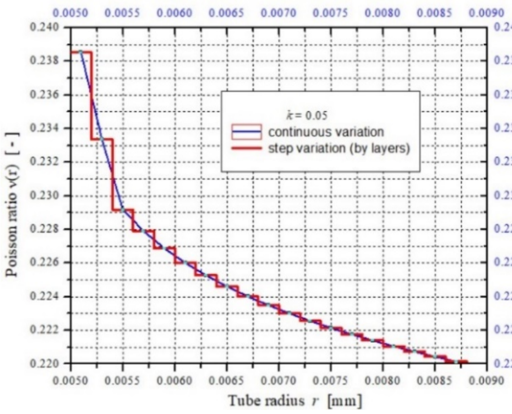


Fig. 7. Poisson's ratio step variation.

TABLE III. EQUIVALENT YOUNG'S MODULUS OF FGM

Young's Modulus [Pa]x10[11]						
k	Layer number					
	5	10	20	30	40	80
0.05	3.661	3.657	3.654	3.654	3.653	3.653
0.10	3.532	3.521	3.521	3.521	3.521	3.519
0.25	3.110	3.184	3.184	3.184	3.184	3.181
0.50	2.781	2.769	2.769	2.769	2.769	2.767
1.00	2.250	2.250	2.250	2.250	2.250	2.250
2.00	1.723	1.733	1.733	1.733	1.733	1.733
4.00	1.299	1.319	1.319	1.319	1.319	1.320
6.00	1.113	1.141	1.141	1.141	1.141	1.143
8.00	1.005	1.042	1.042	1.042	1.042	1.044

The function of this equivalent module varies with the power coefficient and the number of layers. An important problem that arises when using the multilayer wall concept is the choice of the number of layers. As it is not possible to work with an infinite number, a reasonable number must be chosen [15-16], which ensures the required accuracy of the calculus of functionally graded discs. The equivalent modulus of elasticity (Young's modulus) is slightly influenced by the number of layers but strongly determined by the power coefficient *k*. The results of the investigation of the number of layers were published in [16-17], but we also have an available recomandation regarding the layer thickness in (3). Such a value allowed us to obtain acceptable results.

$$th_L \leq h/20\ [m] \tag{3}$$

TABLE IV. EQUIVALENT POISSON RATIO OF FGM

ν_{ech}[-]						
k	Layer number					
	5	10	20	30	40	80
0.05	0.225	0.225	0.225	0.225	0.225	0.225
0.10	0.229	0.230	0.230	0.230	0.230	0.230
0.25	0.241	0.242	0.242	0.242	0.242	0.242
0.50	0.256	0.256	0.257	0.257	0.257	0.257
1.00	0.275	0.275	0.275	0.275	0.275	0.275
2.00	0.294	0.293	0.293	0.293	0.293	0.293
4.00	0.309	0.308	0.308	0.308	0.308	0.308
6.00	0.315	0.315	0.314	0.314	0.314	0.314
8.00	0.319	0.318	0.318	0.318	0.318	0.318

The two concepts utilized, the multilayer wall and the equivalent material, demonstrate the great advantage of either considering the variation of Poisson's ratio or not, in addition to accessibility. Poisson's ratio variation is performed according to the same procedure and methodology presented for Young's modulus, and its equivalent values are depicted in Table IV. The analysis of the data in Table IV shows that the Poisson ratio variation with the number of layers is practically negligible. An important variation exists concerning the power coefficient. The equivalent density can be calculated by:

$$dM(r) = \rho(r) \cdot dV(r) \tag{4}$$

$$\rho(r) = \rho_b + (\rho_t - \rho_b)\left(\frac{r-R_1}{h}\right)^k \tag{5}$$

$$dM = 2\pi\rho_b r dr + 2\pi\frac{(\rho_t-\rho_b)}{h^k}(r-R_1)^k r dr \tag{6}$$

$$M = 2\pi\rho_b\int_{R_1}^{R_2} r dr + 2\pi\frac{(\rho_t-\rho_b)}{h^k}\int_{R_1}^{R_2}(r-R_1)^k r dr \tag{7}$$

$$V = \pi(R_2^2 - R_1^2) \tag{8}$$

$$\rho_{echiv} = \frac{M}{V} \tag{9}$$

Using these equations and the data from the case study for the power law with coefficient $k = 0.05$, the equivalent density (ρ_{ech}) value is 3908.36 kg/m^3. The equivalent density value, as an average of the values of each layer, is 3900.80 kg/m^3 and this differs from the analytical calculus by -0.19%. The value of this error practically validates the proposed multilayer wall concept and gives confidence and safety to the calculation of the rotating disc, made of FGMs, using both concepts.

VI. ANALYTICAL CALCULUS OF FGDS USING EQUIVALENT PROPERTIES.

The analytical calculus follows the general theory of a rotating disc having a uniform thickness [18]. Engaging a polar reference system (r, θ) on the cross-section (the z-axis is the rotating disc axis), where the r direction coincides with the x-axis, the traditional notations from the elasticity theory, and the calculus model displayed in Figure 8. The differential equation of the radial displacements $u=u(r)$ is written as [18]:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\cdot\frac{du}{dr} - \frac{u}{r^2} = -\frac{1-\nu^2}{E}\rho\omega^2 r \tag{10}$$

The solution of this equation is:

$$u(r) = \frac{\rho\omega^2}{8E}\left[\frac{(3+\nu)}{(1-\nu)}(R_1^2 + R_2^2)r + \frac{(3+\nu)}{(1-\nu)}\cdot\frac{R_1^2R_2^2}{r} - (1-\nu^2)r^3\right] \quad (11)$$

In (11), ν can also be a ν_{ech}, an adopted constant value, or a function $\nu(r)$. Table V presents the analytical calculus results, based on an equivalent material concept [9, 16] and using equivalent values for a multilayer wall (20 layers). In (4) and (5), the density is considered constant at the value of the equivalent density. This can be evaluated as the average of density values at the level of lines or layers (Figure 6). Table V shows a relatively low influence of Poisson's ratio and a very good agreement of the results for $\nu = \nu(r)$ for its equivalent value of 0.225.

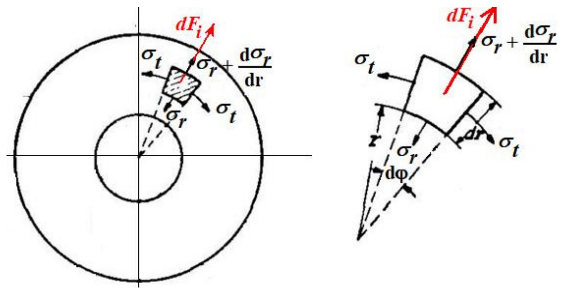


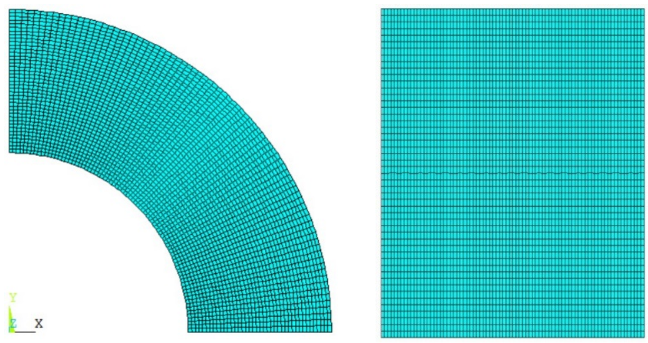
Fig. 8. The calculus model of the disk element equilibrium.

TABLE V. THE RESULTS OF THE ANALYTICAL CALCULUS FOR POWER COEFFICIENT $k = 0.05$

r	ν = 0.30	ν = 0.225	ν(r)
	u(*r*)		
[m]	[m]		
0.0050	1.525E-09	1.543E-09	1.535E-09
0.0052	1.506E-09	1.527E-09	1.524E-09
0.0054	1.489E-09	1.513E-09	1.511E-09
0.0056	1.474E-09	1.501E-09	1.499E-09
0.0058	1.461E-09	1.490E-09	1.489E-09
0.0060	1.449E-09	1.480E-09	1.479E-09
0.0062	1.438E-09	1.472E-09	1.471E-09
0.0064	1.428E-09	1.464E-09	1.464E-09
0.0066	1.419E-09	1.457E-09	1.457E-09
0.0068	1.41E-09	1.451E-09	1.451E-09
0.0070	1.401E-09	1.444E-09	1.445E-09
0.0072	1.394E-09	1.438E-09	1.439E-09
0.0074	1.386E-09	1.433E-09	1.434E-09
0.0076	1.378E-09	1.427E-09	1.429E-09
0.0078	1.371E-09	1.421E-09	1.423E-09
0.0080	1.363E-09	1.415E-09	1.418E-09
0.0082	1.355E-09	1.409E-09	1.412E-09
0.0084	1.347E-09	1.403E-09	1.406E-09
0.0086	1.339E-09	1.396E-09	1.400E-09
0.0088	1.33E-09	1.389E-09	1.393E-09
0.0090	1.321E-09	1.381E-09	1.385E-09

VII. MODELS WITH THE FINITE ELEMENTS OF FUNCTIONALLY GRADED DISCS

Three main finite element models were used. Figures 9 and 10 show the version with 20 material layers. In the performed calculations, these models were utilized in several variants, having 5, 10, 20, 30, and 40 layers. Each time, at least two finite elements were employed for the thickness of one layer. The first finite element model, shown in Figure 9(a), is a 2D model, using PLANE182 element, in-plane stress/strain, from the finite element library of Ansys software. PLANE182 is a finite element with 4 nodes, two degrees of freedom per node, and linear interpolation functions [17, 19-21]. The second FE model, depicted in Figure 9(b), is also a 2D model utilizing the same PLANE182 element [22], but with its option as an axial-symmetric finite element. Figure 10 illustrates the finite element Model-3, which is a 3D one that uses the SOLID185 element. The finite element model is a brick type, with 8 nodes and three degrees of freedom per node [17, 23]. This model is a simplified one, based on the existence of two planes of symmetry [24-25]. The longitudinal dimension is an arbitrary one of 0.005 m.



(a) (b)

Fig. 9. Finite element Model-1 and Model-2.

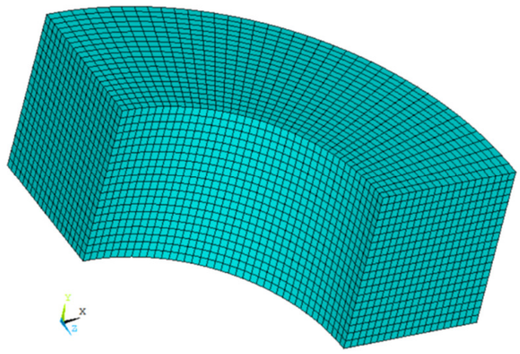


Fig. 10. Finite element Model-3.

VIII. RESULTS OF NUMERICAL CALCULUS BY FEM

The finite element models were organized for the analysis and presentation of the results as follows:

- Model-1: 2D, plane stress, equivalent material
- Model-2: 2D, plane stress, 20 layers, equivalent material
- Model-3: 3D, brick 8 nodes, equivalent material
- Model-4: 2D, axi-simmetric, equivalent material

Table VI entails the findings of the finite element analysis for these four calculus models. Figure 11 illustrates the field of radial displacements in the rotating FGM disc. There is a very good agreement between the models and a very good closeness to the analytically calculated values based on the equivalent material, especially when using the 2D model in the plane stress state. Model-1 and Model-3 showed errors below 1%.

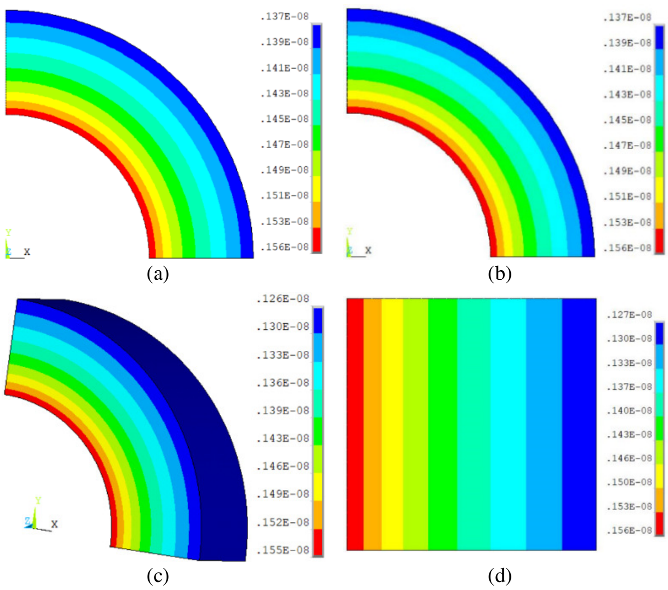


Fig. 11. Finite element models: (a) Model-1, (b) Model-2, (c) Model-3, and (d) Model-4.

TABLE VI. COMPARATIVE RESULTS

Results	w_{R_1} [m]*10^{-9}			w_{R_2} [m]*10^{-9}		
	Analyt.	**FE**	**Err. [%]**	**Analyt.**	**FE**	**Err. [%]**
Model-1	1.568	1.56	-0.51	1.358	1.37	0.73
Model-2		1.56	-0.64		1.37	0.73
Model-3		1.55	-1.27		1.26	-7.35
Model-4		1.56	-0.64		1.27	-6.62

TABLE VII. COMPARATIVE RESULTS

r	$u(r)_{analytic}$	$u(r)_{FEM}$	Errors
[m]	**[m]**	**[m]**	**[%]**
0.0050	1.568E-09	1.560E-09	-0.51
0.0052	1.548E-09	1.538E-09	-0.70
0.0054	1.531E-09	1.522E-09	-0.59
0.0056	1.516E-09	1.509E-09	-0.49
0.0058	1.502E-09	1.496E-09	-0.38
0.0060	1.490E-09	1.486E-09	-0.28
0.0062	1.478E-09	1.476E-09	-0.19
0.0064	1.468E-09	1.467E-09	-0.10
0.0066	1.458E-09	1.458E-09	-0.01
0.0068	1.449E-09	1.451E-09	0.07
0.0070	1.441E-09	1.443E-09	0.16
0.0072	1.433E-09	1.436E-09	0.23
0.0074	1.425E-09	1.429E-09	0.31
0.0076	1.417E-09	1.422E-09	0.37

Table VII compares the outcomes of the analytical calculation with those of the numerical one utilizing Model-1. The comparative analysis of the numerical results practically validates the two calculus concepts (multilayer wall and equivalent material). From the data in Tables VI and VII, a series of useful conclusions arise, both for the analysis of the rotating FGM disc and that of the finite element modeling. Between those four considered models, the radial displacement values are practically the same. Therefore, any of these four models can be applied in practice. Moreover, the results of the numerical calculus with FE are extremely close to those of the analytical calculus, based on the equivalent material concept. This calculus concept is easier to use and leads to closer findings to the ones of the analytical solution.

IX. CONCLUSIONS

This study aimed to provide methods and models for the calculus of rotating discs made of FGMs, which should be accessible, precise enough, and as operational as possible. Along with other structural elements, such as plates and beams, rotating discs are widely used, and their realization from FGMs can lead to special technical performances. The calculus of a rotating disc made of FGMs was based on the concepts of multilayer wall and equivalent material. Both calculus concepts can be employed in numerical analysis by the FE method. Any elastic or physical properties of the material can be implemented in the equivalent material (Young's modulus, Poisson's ratio, density). The use of the multilayer material concept allows us to consider the simultaneous variation of all the material characteristics in the case of numerical calculus utilizing the finite element method. The theoretical concepts of approaching the calculus of the rotating disc made of FGMs were practically exemplified in a case study to make possible the substantiation of the conclusions and observations through quantitative arguments.

The material law considered in the case study was the power law, which is one of the most common laws involved in the manufacturing of FGMs. The power coefficient usual value is a factor that favors the imposition of superior properties of the functional component materials, but in all cases, a study was carried out regarding its influence on the results. Every time and in all the situations studied, the power factor impact was significant, which substantially determined the behavior of the rotating disc under load. The load taken into account in this paper was only the inertial force as a result of the disk rotation, although the same method can be used for internal and/or external pressure loading. The study considered the multilayer wall to have 5, 10, 20, 30, 40, and 80 layers. The accuracy of the results was fully consistent with the characteristics of the finite element method. The analytical calculus presented, based on the concept of equivalent material (homogeneous and isotropic material with equivalent characteristics), represents a valid way of solving similar problems.

Both analytical calculations based on the concept of equivalent material as well as the numerical calculation using the finite element method can acknowledge the simultaneous variation of the Poisson ratio on the wall thickness. This study findings confirm the statement that the influence of a variable Poisson ratio is small, but complete neglection in any situation is debatable. The use of such discs is varied and a high precision of the calculation is absolutely necessary, therefore, this may require contemplating the variation of the Poisson ratio. Neglecting the variation of Poisson's ratio mainly brings convenience to the mathematical calculation, but can sometimes lead to unwanted errors. Finally, a general conclusion regarding this study can be that the methods and models presented are operational, accessible, can be supported by usual structural analysis programs through the finite element method, and have a high degree of generality.

REFERENCES

[1] F. Tarlochan, "Functionally Graded Material: A New Breed of Engineered Material," *Journal of Applied Mechanical Engineering*, vol. 2, no. 2, 2013, https://doi.org/10.4172/2168-9873.1000e115.

[2] I. M. El-Galy, B. I. Saleh, and M. H. Ahmed, "Functionally graded materials classifications and development trends from industrial point of view," *SN Applied Sciences*, vol. 1, no. 11, Oct. 2019, Art. no. 1378, https://doi.org/10.1007/s42452-019-1413-4.

[3] A. Toudehdehghan, J. W. Lim, K. E. Foo, M. I. N. Ma'arof, and J. * Mathews, "A brief review of functionally graded materials," *MATEC Web Conferences*, vol. 131, 2017, https://doi.org/10.1051/matecconf/201713103010.

[4] A. Hadj Mostefa, S. Merdaci, and N. Mahmoudi, "An Overview of Functionally Graded Materials «FGM»," in *Proceedings of the Third International Symposium on Materials and Sustainable Development*, 2018, pp. 267–278, https://doi.org/10.1007/978-3-319-89707-3_30.

[5] Y. Z. Chen and X. Y. Lin, "Elastic analysis for thick cylinders and spherical pressure vessels made of functionally graded materials," *Computational Materials Science*, vol. 44, no. 2, pp. 581–587, Dec. 2008, https://doi.org/10.1016/j.commatsci.2008.04.018.

[6] Y. Z. Chen and X. Y. Lin, "An alternative numerical solution of thick-walled cylinders and spheres made of functionally graded materials," *Computational Materials Science*, vol. 48, no. 3, pp. 640–647, May 2010, https://doi.org/10.1016/j.commatsci.2010.02.033.

[7] M. Akbarzadeh Khorshidi and D. Soltani, "Analysis of functionally graded thick-walled cylinders with high order shear deformation theories under non-uniform pressure," *SN Applied Sciences*, vol. 2, no. 8, Jul. 2020, Art. no. 1362, https://doi.org/10.1007/s42452-020-3179-0.

[8] V. Năstăsescu and S. Marzavan, "An Overview of Functionally Graded Material Models," *Proceedings of the Romanian Academy*, vol. 23, no. 3, pp. 259–267, 2022.

[9] S. Marzavan and V. Nastasescu, "Displacement calculus of the functionally graded plates by finite element method," *Alexandria Engineering Journal*, vol. 61, no. 12, pp. 12075–12090, Dec. 2022, https://doi.org/10.1016/j.aej.2022.06.004.

[10] S. Marzavan and V. Nastasescu, "Free Vibration Analysis of a Functionally Graded Plate by Finite Element Method," *Ain Shams Engineering Journal*, vol. 14, no. 8, Aug. 2023, Art. no. 102024, https://doi.org/10.1016/j.asej.2022.102024.

[11] V. Năstăsescu, "The influence of Poisson's ratio in the calculus of functionally graded plates," *Journal of Engineering Sciences and Innovation*, vol. 7, no. 4, pp. 393–402, Dec. 2022, https://doi.org/10.56958/jesi.2022.7.4.393.

[12] D. G. Zisopol, M. Minescu, and D. V. Iacob, "A Theoretical-Experimental Study on the Influence of FDM Parameters on the Dimensions of Cylindrical Spur Gears Made of PLA," *Engineering, Technology & Applied Science Research*, vol. 13, no. 2, pp. 10471–10477, Apr. 2023, https://doi.org/10.48084/etasr.5733.

[13] D. G. Zisopol, M. Minescu, and D. V. Iacob, "A Theoretical-Experimental Study on the Influence of FDM Parameters on the Dimensions of Cylindrical Spur Gears Made of PLA," *Engineering, Technology & Applied Science Research*, vol. 13, no. 2, pp. 10471–10477, Apr. 2023, https://doi.org/10.48084/etasr.5733.

[14] D. G. Zisopol, M. Minescu, M. Badicioiu, and M. M. Caltaru, "Theoretical and Experimental Investigations on 20 Inches Threaded Casing Connections Failure under Field Conditions," *Engineering, Technology & Applied Science Research*, vol. 11, no. 4, pp. 7464–7468, Aug. 2021, https://doi.org/10.48084/etasr.4318.

[15] V. Nastasescu, "The Using of the Multilayer Plate Concept in the Calculus of Functionally Graded Plates," *Applied Sciences*, vol. 12, no. 21, 2022, https://doi.org/10.3390/app122110695.

[16] A. Toma, F. Dragoi, and O. Postavaru, "Enhancing the Accuracy of Solving Riccati Fractional Differential Equations," *Fractal and Fractional*, vol. 6, no. 5, May 2022, Art. no. 275, https://doi.org/10.3390/fractalfract6050275.

[17] "Theory Reference for the Mechanical APDL and Mechanical Applications," ANSYS Inc, Canonsburg, PA, USA, Apr. 2009.

[18] S. Timoshenko and J. N. Goodier, *Theory of Elasticity*. New York, NY, US: McGraw-Hill, 1969.

[19] K.-J. Bathe, *Finite Element Procedures*. Hoboken, NJ, USA: Prentice Hall, 1996.

[20] K. H. Huebner, E. A. Thornton, and T. G. Byrom, *The Finite Element Method for Engineers, 3rd Edition*, 3rd edition. New York, NY, USA: Wiley-Interscience, 1994.

[21] T. J. R. Hughes, The Finite Element Method: Linear Static and Dynamic Finite Element Analysis. Mineola, NY, USA: Dover Publications, 2012.

[22] M. Stan and D. G. Zisopol, "Modeling and Optimization of Piston Pumps for Drilling," *Engineering, Technology & Applied Science Research*, vol. 13, no. 2, pp. 10505–10510, Apr. 2023, https://doi.org/10.48084/etasr.5714.

[23] A. Toma, "Generating Functions for the Mean Value of a Function on a Sphere and Its Associated Ball in," *Journal of Inequalities and Applications*, vol. 2008, no. 1, 2008, Art. no. 656329, https://doi.org/10.1155/2008/656329.

[24] O. C. Zienkiewicz and R. L. Taylor, *The Finite Element Method: Basic Formulation and Linear Problems*, Subsequent edition. London, UK: McGraw-Hill College, 1987.

[25] M. Tanase, D. G. Zisopol, and A. I. Portoaca, "A Study regarding the Technical-Economical Optimization of Structural Components for enhancing the Buckling Resistance in Stiffened Cylindrical Shells," *Engineering, Technology & Applied Science Research*, vol. 13, no. 5, pp. 11511–11516, Oct. 2023, https://doi.org/10.48084/etasr.6135.