Investigating the Response Variability of Statically Determined Sandwich Beams considering two Random Fields of Elastic Modulus

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ABSTRACT

In this paper, the displacement variation in sandwich beams is determined by employing a semi-analytical approach. The classical displacement is calculated by integration using Mohr's equation, although the integration is complicated due to the inclusion of random fields in the inertial moment term. Using the trapezoidal rule to compute these integrals, the random fields are discretized into random variables at the nodal point of the beam segments. Thus, the expected displacement, standard deviation, and coefficient of variation can be computed. To validate the results, the random fields are simulated using a previously described spectral method. The results of numerical examples were compared with the semi-analytical method and the Monte Carlo simulation demonstrating the high accuracy of the proposed method. The results also illustrate the influence of the parameters of the random fields of elastic modulus on the variability of displacement.

Keywords-sandwich beam; random field; semi-analytical approach; MSC

I. INTRODUCTION

The main advantage of sandwich beams is their high stiffness-to-bending weight ratio, enabling numerous applications in automobile industry [1, 2] and structural engineering [3, 4]. Researchers who study the analysis of structural elements, such as beams, frames, piles, and plates, have the option to employ various methods, including analytical [5-10], semi-analytical [11, 12], and numerical techniques such as the Finite Element Method (FEM) [13-15]. For example, authors in [16] analyzed sandwich beams with functionally graded materials using the scaled boundary FEM. In reality, many input parameters in structural calculations are often assumed to be random variables. The random oscillation of a moving load on a rough road surface has been investigated according to international standards using Matlab in [17]. In [18], the variation of deformation of non-uniform columns has been studied using the stochastic FEM with a weighted technique to discretize a random field. Authors in [19] developed stochastic finite elements using the point method to discretize a random field and determine the critical loads of non-uniform columns. Authors in [20] applied Karhunen–Loéve expansion to develop the stochastic FEM for prestressed beams and frames [20]. In [21], the variation in the natural frequency of a beam was computed using stochastic FEM with a random field approach for the elastic modulus.

For sandwich structures, many studies addressed various types of mechanic problems, including static, dynamic, stability, and random problems. Authors in [22] used the R-functions method to investigate the critical loads on a sandwich plate subjected to non-uniform loading. Authors in [23] investigated the natural frequencies of functionally graded sandwich beams while considering trigonometric shear deformation. Authors in [24] developed the hybrid stochastic time-domain spectral element method for higher-order sandwich beams using the Karhunen–Loéve expansion. The transient responses of a sandwich beam on a spring support under a moving load have also been investigated with FEM [25]. Although various studies have considered random parameters in sandwich beams, all of them used the FEM, and to the best of our knowledge, there are no reports using a semi-analytical approach. Despite numerous contributions to problems involving random parameters, there have been no reported calculations for a sandwich beam using a classical method for determining displacement and considering random fields in the elastic modulus of multiple material layers.

This paper investigates the displacement variability of a statically determined sandwich beam subjected to a distributed load, considering two random fields of elastic modulus, using a semi-analytical method. Calculating variations of displacements and internal forces at an advanced level will be used to evaluate the reliability of the structures.

II. DISPLACEMENT OF A SANDWICH BEAM COMPUTED BY THE CLASSICAL APPROACH

Consider the simply supported sandwich beam subjected to a uniformly distributed load, as shown in Figure 1.

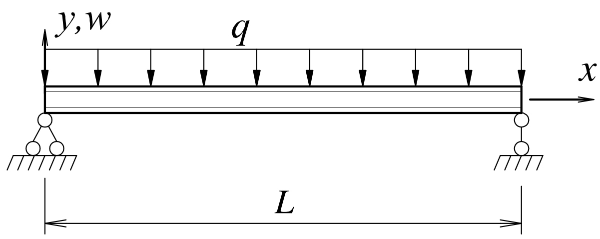

Fig. 1. A simply supported sandwich beam subjected to a uniformly distributed load

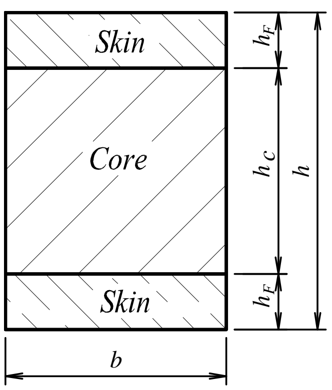

Fig. 2. Section of the sandwich beam.

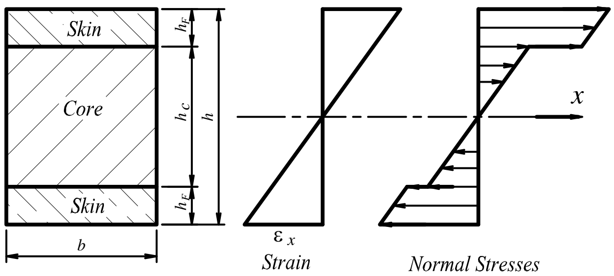

Fig. 3. Strain and normal stress distributions on a sandwich beam.

$$U_e = \frac{1}{2}\int_0^{L_e} \left\{ \begin{array}{l} \underbrace{\int_A \left\{ E_{Face} \left[y \frac{\partial^2 w}{\partial x^2} \right]^2 \right\} dA}_{Face} \\ + \underbrace{\int_A \left\{ E_{Core} \left[y \frac{\partial^2 w}{\partial x^2} \right]^2 \right\} dA}_{Core} \end{array} \right\} dx \quad (1)$$

The displacement at the middle of the beam using Mohr's equation from classical mechanics is expressed as [26]:

$$w_m = \int_0^{L/2} \frac{M(x)\bar{M}(x)}{EI_{eq}} dx + \int_{L/2}^{L} \frac{M(x)\bar{M}(x)}{EI_{eq}} dx \quad (2)$$

The equivalent flexural rigidity is described by:

$$EI_{eq} = \frac{E_c b h_c^3}{12} + 2E_F b \left[\frac{h_F^3}{12} + \left(\frac{h}{2} - \frac{h_F}{2} \right)^2 \right] \quad (3)$$

The flexural diagram of a simple sandwich beam in both its actual and virtual states is shown in Figure 4.

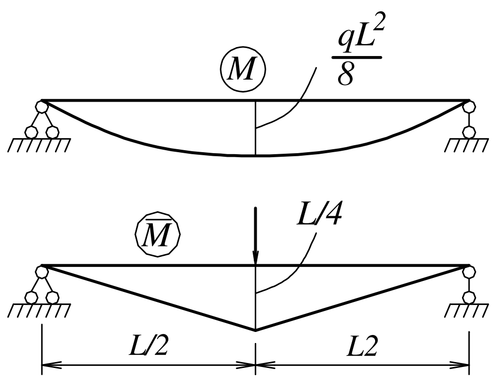

Fig. 4. Bending moment diagrams of a sandwich beam in the actual (top) and virtual (bottom) states.

The moment expressions for a sandwich beam are expressed by:

- Actual states:

$$M = \frac{qL}{2}x - \frac{qx^2}{2} \quad (4)$$

- Virtual states:

$$\bar{M}=\begin{cases}\dfrac{x}{2} \text{ if } x\le \dfrac{L}{2}\\ \dfrac{L-x}{2} \text{ if } x\ge \dfrac{L}{2}\end{cases} \tag{5}$$

III. SEMI-ANALYTICAL SOLUTION

The random fields of the elastic moduli in the skin and core are assumed to be one-dimensional and homogeneous:

$$\begin{aligned}E_c(x)&=\left[1+r_c(x)\right]E_{c0}\\ E_F(x)&=\left[1+r_F(x)\right]E_{F0}\end{aligned} \tag{6}$$

where $r_c(x)$, $r_F(x)$ are one-dimensional homogeneous random fields with a mean of zero. Using the coefficients of variation σ_c, σ_F and correlation distances d_c, d_F for the random fields $r_c(x)$, $r_F(x)$, the autocorrelation functions of these random fields are described by:

$$\begin{aligned}R_c(\xi)&=\sigma_c^2\exp\left(-\frac{|\xi|}{d_c}\right)\\ R_F(\xi)&=\sigma_F^2\exp\left(-\frac{|\xi|}{d_F}\right)\end{aligned} \tag{7}$$

where $\xi=x_j-x_i$ is the relative distance vector.

The random fields of the elastic moduli in the beam are approximated by random variables r_i at n points in the element:

$$EI_{eq}=EI_0+\alpha r_c(x)+\beta r_{F(x)} \tag{8}$$

where:

$$\begin{aligned}EI_0&=\frac{E_{c0}bh_c^3}{12}+2E_{F0}b\left[\frac{h_F^3}{12}+h_F\left(\frac{h}{2}-\frac{h_F}{2}\right)^2\right]\\ \alpha&=\frac{E_{c0}bh_c^3}{12}\\ \beta&=2E_{F0}b\left[\frac{h_F^3}{12}+h_F\left(\frac{h}{2}-\frac{h_F}{2}\right)^2\right]\end{aligned} \tag{9}$$

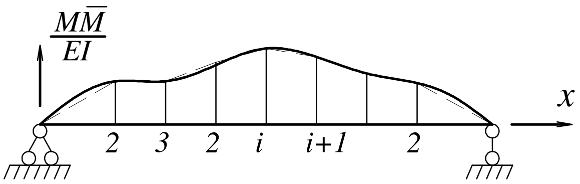

Fig. 5. Randomness in the elastic modulus of the beam.

The displacement is approximated by:

$$w_m=\frac{1}{2}\sum\Delta x\left\{\frac{M_i(x)\bar{M}_i(x)}{\left(EI_{eq}\right)_i}+\frac{M_{i+1}(x)\bar{M}_{i+1}(x)}{\left(EI_{eq}\right)_{i+1}}\right\} \tag{10}$$

Ignoring the infinitesimal term of higher order, displacement is approximated using the following expression:

$$\begin{aligned}w_m&\approx\sum\frac{M_i(x)\bar{M}_i(x)}{\left(EI_{eq}\right)_i}\Delta x\\ &\approx\sum\frac{M_i(x)\bar{M}_i(x)}{EI_0}\left[1-\alpha\frac{r_c(x)}{EI_0}-\beta\frac{r_{F(x)}}{EI_0}\right]\Delta x\end{aligned} \tag{11}$$

Then, the expected displacement is:

$$\begin{aligned}\bar{W}&=E\left\{\sum\frac{M_i(x)\bar{M}_i(x)}{EI_0}\left[1-\alpha\frac{r_c(x)}{EI_0}-\beta\frac{r_{F(x)}}{EI_0}\right]\Delta x\right\}\\ &=\sum\left\{\frac{M_i(x)\bar{M}_i(x)}{EI_0}\right\}\end{aligned} \tag{12}$$

The variance of the displacement is:

$$\begin{aligned}&\operatorname{var}(w_m)=\\ &=E\left\{\begin{aligned}&\left(\sum\frac{M_i(x)\bar{M}_i(x)}{EI_0}\left[1-\alpha\frac{r_c(x)}{EI_0}-\beta\frac{r_{F(x)}}{EI_0}\right]\Delta x-\bar{W}\right)\times\\ &\left(\sum\frac{M_i(x)\bar{M}_i(x)}{EI_0}\left[1-\alpha\frac{r_c(x)}{EI_0}-\beta\frac{r_{F(x)}}{EI_0}\right]\Delta x-\bar{W}\right)\end{aligned}\right\}\end{aligned} \tag{13}$$

The coefficient of variation (COV) for the displacement is:

$$\mathrm{COV}=\frac{\sqrt{\operatorname{var}(w_m)}}{\bar{W}} \tag{14}$$

IV. MONTE CARLO SIMULATION

The previous section presented the semi-analytical approach used for determining the mean and variance of displacements. However, it is essential to validate the results of this approach. Therefore, Monte Carlo simulation is conducted using the spectral representation method [27]. In this, the random field of the elastic modulus is generated by:

$$\begin{aligned}r_c(x)&=\sqrt{2}\sum_{n=0}^{N-1}A_n^c\cos\left(\omega_n^c x+\phi_n^c\right)\\ r_E(x)&=\sqrt{2}\sum_{n=0}^{N-1}A_n^F\cos\left(\omega_n^F x+\phi_n^F\right)\end{aligned} \tag{15}$$

$$\begin{aligned}A_n^F&=\sqrt{2S_{ff}^F\left(\omega_n^F\right)\Delta\omega^F}\\ A_n^c&=\sqrt{2S_{ff}^c\left(\omega_n^c\right)\Delta\omega^c}\\ \Delta\omega^F&=\frac{\omega_u^F}{N};\ \Delta\omega^c=\frac{\omega_u^c}{N}\\ \omega_n^F&=n\Delta\omega^F,\ \omega_n^c=n\Delta\omega^c,\ n=0,1,2,...,N-1\end{aligned} \tag{16}$$

where ω_n^F, ω_n^c denote the upper cut-off frequency for the skin and core of a sandwich beam, beyond which the power spectral density functions $S_{ff}^F\left(\omega_n^F\right), S_{ff}^c\left(\omega_n^c\right)$ denote the random field of stiffness for the elastic moduli in the skin and core of a sandwich beam, respectively.

By substituting (15) into (11) the displacement can be evaluated by:

$$w_m \approx$$
$$\approx \sum \frac{M_i(x)\bar{M}_i(x)}{EI_0}\left[\begin{array}{l} 1-\dfrac{\alpha}{EI_0}\left\{\sqrt{2}\sum\limits_{n=0}^{N-1} A_n^c \cos\left(\omega_n^c x+\phi_n^c\right)\right\} \\ -\dfrac{\beta}{EI_0}\left\{\sqrt{2}\sum\limits_{n=0}^{N-1} A_n^F \cos\left(\omega_n^F x+\phi_n^F\right)\right\} \end{array}\right]\Delta x \quad (17)$$

V. NUMERICAL EXAMPLES

A sandwich beam subjected to uniformly distributed loads, is considered for the evaluation of the semi-analytical approach (Figure 1). The sandwich beam has a length of L = 10 m and height of h = 60 cm, with h_F = 2 cm and h_c = 56 cm. The material properties of the beam are given in Table I.

TABLE I. MATERIAL PROPERTIES OF THE SANDWICH BEAM

Fiber	Material	Young's modulus (GPa)	Coefficient of variation
Skin layer	Steel	200	0.1
Core	Concrete	30	0.1

The sandwich beam is divided into 20 segments, and variations in the two random fields are considered while maintaining an equal coefficient of variation $\sigma_c = \sigma_F$ and equal correlation length distance $d_c = d_F = d(m)$. In Figure 6, we compare the coefficient of variation for the displacements of the sandwich beam obtained using the semi-analytical method (the proposed approach) and the Monte Carlo simulation. The variability coefficient results of the semi-analytical method are consistent with the numerical Monte Carlo simulation.

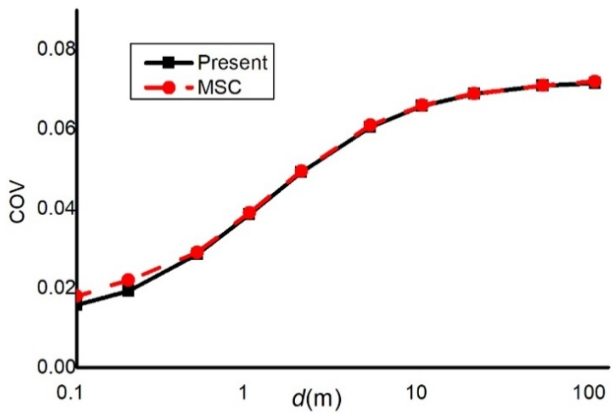

Fig. 6. Coefficients of variation for the beam displacement determined with the proposed method and the Monte Carlo simulation.

As the next step, different cases are employed to investigate the influence of random field parameters for the elastic moduli in the two materials on the displacement variability coefficient. Three cases of the coefficient of variation for the two random fields of the elastic moduli for the skin and core layers are considered, as shown in Table II.

Figure 7 illustrates the variability coefficient of the beam displacement corresponding to three cases of variation in the elastic moduli of the material properties, showing that the coefficient of variation increases with the correlation length. In general, the curve of the coefficient of variation for the displacement in case 1, corresponding to the smallest input value of the coefficient of variation, exhibits the smallest coefficient of variation. Additionally, the curve of the coefficient of variation for the displacement in case 3, corresponding to the largest input value of the coefficient of variation, shows the largest coefficient of variation. Thus, the coefficient of variation for the displacement is directly proportional to the coefficient of variation for random elastic modulus.

TABLE II. COEFFICIENT OF VARIATION FOR THE ELASTIC MODULUS

Case	Skin layer	Core
1	0.05	0.10
2	0.10	0.10
3	0.10	0.05

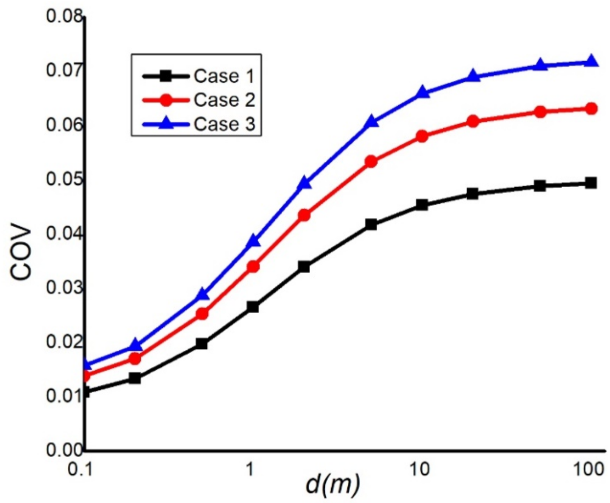

Fig. 7. Coefficients of variation for the beam displacement, determined with the proposed method, considering three cases with different material properties.

VI. CONCLUSIONS

This paper demonstrates a semi-analytical solution for calculating the deflection variation of a statically determined simply supported sandwich beam subjected to a uniformly distributed load. Two random fields of the elastic modulus are discretized by random variables at the beam division nodes. Using Mohr's equation to calculate displacement, an approximate expression for the mean and variance of the displacement is determined. The numerical results are validated with Monte Carlo simulations, demonstrating the accuracy of the proposed semi-analytical approach. Numerical examples illustrate the significant impact of the correlation length distance on the variability coefficient of the displacement. The coefficient of variation of the displacement when the distance is correlated to infinity is smaller than the largest number of the input coefficient of variation of the elastic modulus in the two material layers.

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