Finite Element Analysis of a Double Beam connected with Elastic Springs

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ABSTRACT

This paper develops a finite element method for double beams subjected to static loading. The double beam consists of two Euler–Bernoulli beams connected continuously by an elastic spring connection. The finite element for a double beam is formulated with eight degrees of freedom based on the Euler-Bernoulli beam theory. The finite element method is implemented in MATLAB software to analyze the behavior of the double beams. The MATLAB code calculates the displacements of both the upper and lower beams. Numerical examples are compared with the analytical solution to demonstrate the high accuracy of the proposed method.

Keywords-double beam; FEM; elastic springs

I. INTRODUCTION

Common structural forms in civil engineering and mechanical, such as beams [1-5], frames [6, 7], plate and shell structures [8-15], have garnered significant attention from the researchers. Problems related to beams and frames are significant in various engineering fields such as mechanical and civil engineering, for example, in the design of bridges, and railway systems. The beams rest on a foundation subjected to static or dynamic loads have been extensively studied for many years with various models foundations, including the Winkler foundation model [16], the two-parameter Pasternak foundation model [17], and the viscoelastic foundation model [1]. Authors in [18] improved the two-parameter foundation model based on the theory of Winkler-Zimmerman and Vlasov-Leontyev to calculate the internal forces of the beam resting on the elastic foundation with two parameters using the finite element method.

In engineering, there are structures composed of two layers of materials bonded together with a thin adhesive layer or some simple connections. These structures can be approximated and analyzed as composite beams with resilient or nonlinear connections. For example, a composite beam model can be applied to approximate the behavior of sandwich beams [3, 19]. Authors in [20] investigated accurate solutions for the dynamic problem using the improved Wittrick-Williams algorithm and the stiffness matrix method. The dynamic stiffness method and the Wittrick-Williams algorithm to calculate the dynamics of a double-beam system with a viscoelastic layer were employed in [21]. The stress and strains in the multilayered beam are solved analytically in [22] in which the assumption of the interaction of layers is nonideal. Authors in [23] used an analytical method with the state function to solve the oscillation problem of a double beam system with a viscoelastic intermediate layer. Authors in [24] calculated the oscillations of a double beam with concentrated masses using an analytical approach. Authors in [25] obtained analytical solutions for the dynamics of a double Euler-Bernoulli beam system subjected to moving loads using trigonometric series. The dynamic response of a double-beam subjected to moving force was investigated using analytical methods [26] considering various symmetric boundary conditions. The buckling analysis of composite rods made of three timber layers with consideration for the nonlinear shear deformation of the timber layers is researched in [27] using a semi-analytical method. There have been numerous studies on the behavior of double beams, but most focus on dynamic analysis using analytical methods. This paper performs finite element analysis for double beams with an elastic layer under static loading conditions. The computed results of the proposed approach are compared with those obtained from the analytical solutions.

II. FINITE ELEMENT FORMULATION FOR DOUBLE-BEAMS

Let us consider a double beam consisting of two beams connected through an elastic layer with stiffness k_w, as shown in Figure 1. The double beam element is shown in Figure 2. The displacement fields are approximated by interpolation functions as follows:

$$\begin{cases} w_{Upper}(x) = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4 \\ w_{Lower}(x) = N_1 u_5 + N_6 u_6 + N_3 u_7 + N_4 u_8 \end{cases} \quad (1)$$

The interpolation functions are defined as:

$$\begin{cases} N_3 = 3\dfrac{x^2}{L^2} - 2\dfrac{x^3}{L^3} & N_1 = 1 - 3\dfrac{x^2}{L^2} + 2\dfrac{x^3}{L^3} \\ N_4 = x\left(-\dfrac{x}{L} + \dfrac{x^2}{L^2}\right) & N_2 = x\left(1 - 2\dfrac{x}{L} + \dfrac{x^2}{L^2}\right) \end{cases} \quad (2)$$

The displacement vector of finite element:

$$\{q\}_e = \{u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \quad u_6 \quad u_7 \quad u_8\}^T \quad (3)$$

The stiffness matrix for the upper and bottom beam:

$$[K]_e^1 = \int_{V_{e1}} [B]^T [D][B] dV_1 = E \int_{A_1} y^2 dA \int_0^L [N'']^T [N''] dx \quad (4)$$

$$[K]_e^2 = \int_{V_{e2}} [B]^T [D][B] dV_2 = E \int_{A_2} y^2 dA \int_0^L [N'']^T [N''] dx \quad (5)$$

The stiffness matrix of the element is shown in (6):

$$[K]_e = \begin{bmatrix} \frac{12EI_1}{L^3} & \frac{6EI_1}{L^2} & -\frac{12EI_1}{L^3} & \frac{6EI_1}{L^2} & 0 & 0 & 0 & 0 \\ \frac{6EI_1}{L^2} & \frac{4EI_1}{L} & -\frac{6EI_1}{L^2} & \frac{2EI_1}{L} & 0 & 0 & 0 & 0 \\ -\frac{12EI_1}{L^3} & -\frac{6EI_1}{L^2} & \frac{12EI_1}{L^3} & -\frac{6EI_1}{L^2} & 0 & 0 & 0 & 0 \\ \frac{6EI_1}{L^2} & \frac{2EI_1}{L} & -\frac{6EI_1}{L^2} & \frac{4EI_1}{L} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{12EI_2}{L^3} & \frac{6EI_2}{L^2} & -\frac{12EI_2}{L^3} & \frac{6EI_2}{L^2} \\ 0 & 0 & 0 & 0 & \frac{6EI_2}{L^2} & \frac{4EI_2}{L} & -\frac{6EI_2}{L^2} & \frac{2EI_2}{L} \\ 0 & 0 & 0 & 0 & -\frac{12EI_2}{L^3} & -\frac{6EI_2}{L^2} & \frac{12EI_2}{L^3} & -\frac{6EI_2}{L^2} \\ 0 & 0 & 0 & 0 & \frac{6EI_2}{L^2} & \frac{2EI_2}{L} & -\frac{6EI_2}{L^2} & \frac{4EI_2}{L} \end{bmatrix} \quad (6)$$

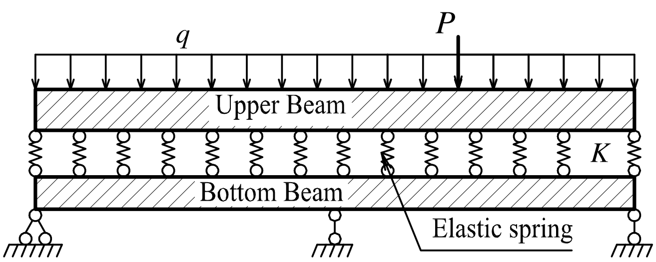

Fig. 1. Double beam.

The potential energy of an elastic layer of element is:

$$U_w = \int_0^{L_e} \frac{1}{2} k_w (w_1 - w_2)^2 dx = \int_0^{L_e} \frac{1}{2} k_w \left([N]\{U\}_1 - [N]\{U\}_2\right)^2 dx \quad (7)$$

The stiffness matrix of an elastic layer is:

$$K_w = k_w \int_0^{L_e} \begin{Bmatrix} [N] \\ -[N] \end{Bmatrix} \{[N] \quad -[N]\} dx \quad (8)$$

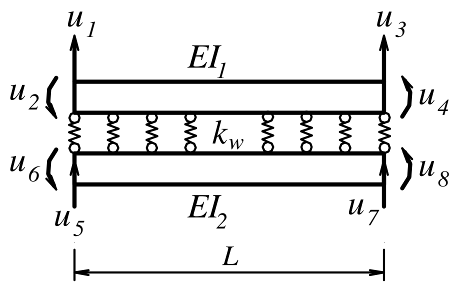

Fig. 2. Double beam element.

$$[K]_w = k_w \begin{bmatrix} \frac{13L}{35} & \frac{11L^2}{210} & \frac{9L}{70} & -\frac{13L^2}{420} & -\frac{13L}{35} & -\frac{11L^2}{210} & -\frac{9L}{70} & \frac{13L^2}{420} \\ & \frac{L^3}{105} & \frac{13L^2}{420} & -\frac{L^3}{140} & -\frac{11L^2}{210} & -\frac{L^3}{105} & -\frac{13L^2}{420} & \frac{L^3}{140} \\ & & \frac{13L}{35} & -\frac{11L^2}{210} & -\frac{9L}{70} & -\frac{13L^2}{420} & -\frac{13L}{35} & \frac{11L^2}{210} \\ & & & \frac{L^3}{105} & \frac{13L^2}{420} & \frac{L^3}{140} & \frac{11L^2}{210} & -\frac{L^3}{105} \\ & & & & \frac{13L}{35} & \frac{11L^2}{210} & \frac{9L}{70} & -\frac{13L^2}{420} \\ & & & & & \frac{L^3}{105} & \frac{13L^2}{420} & -\frac{L^3}{140} \\ & Sym & & & & & \frac{13L}{35} & -\frac{11L^2}{210} \\ & & & & & & & \frac{L^3}{105} \end{bmatrix} \quad (9)$$

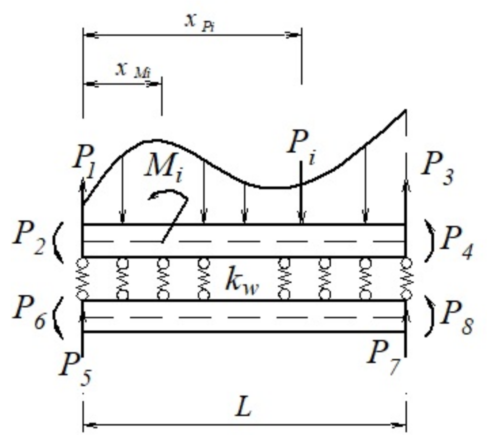

Fig. 3. Loads on double beam element.

The force vectors on the element for distributed loads on the upper beam are:

$$\{P\}_e = \int_0^L \begin{bmatrix} 1-3\frac{x^2}{L^2}+2\frac{x^3}{L^3} \\ x-2\frac{x^2}{L}+\frac{x^3}{L^2} \\ 3\frac{x^2}{L^2}-2\frac{x^3}{L^3} \\ -\frac{x^2}{L}+\frac{x^3}{L^2} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} p_0 dx = \begin{Bmatrix} \frac{p_o L}{2} \\ \frac{p_0 L^2}{12} \\ \frac{p_o L}{2} \\ -\frac{p_0 L^2}{12} \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (10)$$

III. NUMERICAL EXAMPLES

A. *Example 1*

The first example was conducted to verify the results of the finite element method presented above with the analytical solution [28]. In this section, the simply supported double beam is composed using two rectangular cross-section beams connected through an elastic spring with a length of 10 m, subjected to a uniformly distributed load of q = 10 kN/m.

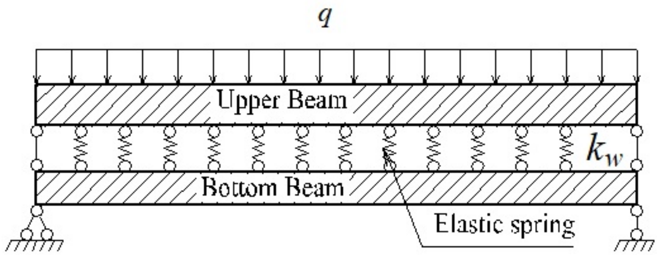

Fig. 4. Simply supported double beam.

Comparing the results obtained from the finite element method and the analytical method, the difference is usually very small, indicating good agreement between the two approaches.

TABLE I. DISPLACEMENTS AT THE MIDLE BEAM

Stiffness k (N/m)	Upper beam			Bottom beam		
	Present (FEM)	Analytic solution [28]	Error (%)	Present (FEM)	Analytic solution [28]	Error (%)
1E+05	33.375	33.375	0	8.2917	8.2918	1.2E-3
1E+06	23.5322	23.522	0	18.1446	18.1447	5.5E-4

Figure 5 shows the shear force and bending moment diagrams for the upper and bottom beam. The upper beam subjected to a direct load lead to significant bending moment, while the bottom beam is smaller bending moment. Conversely, the shear forces between the upper and bottom beams show minimal difference.

Figure 6 illustrates the displacements of the upper and bottom beams with varying stiffness of the elastic springs. The results show that as the stiffness increases, the difference in displacement between the two beams decreases gradually. The results show that the interaction between the two beams depends significantly on the stiffness of the elastic layer. As the stiffness of the elastic layer increases, more load is transmitted from the upper to the lower beam, resulting in a decrease in the displacement of the upper beam and an increase in the displacement of the lower beam. When the stiffness of the elastic layer becomes very high, the displacements of both beams become equal.

(a)

(b)

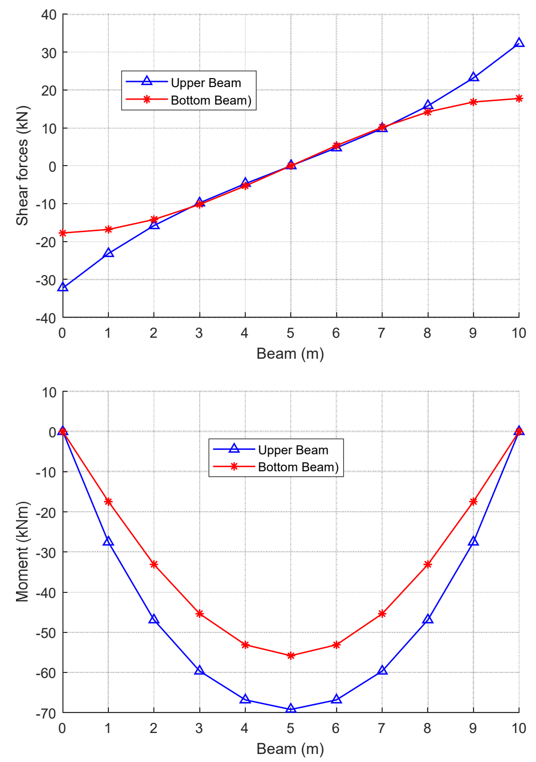

Fig. 5. Double beam: Free upper and botom beams. (a) Shear forces, (b) moment.

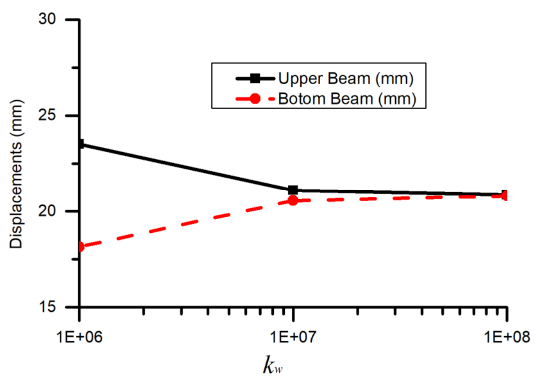

Fig. 6. Displamement of beam and stiffness of the elastic layer.

B. *Example 2*

The double beam is composed using two rectangular cross-section beams connected through an elastic spring with a length of 10 m, subjected to a concentrated load P = 10 kN, as shown in Figure 7.

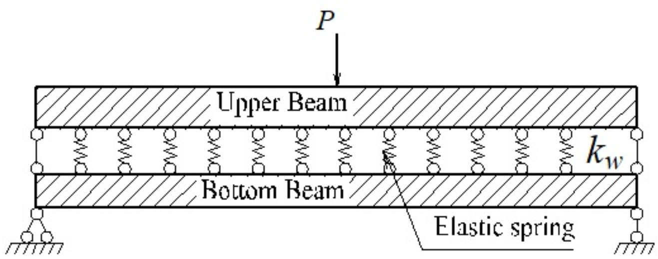

Fig. 7. Double beam: Free upper beam and bottom beam.

(a)

(b)

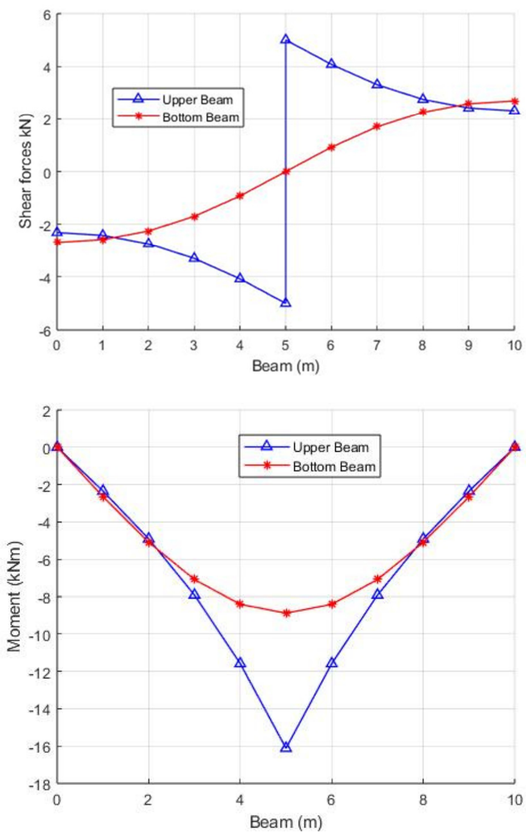

Fig. 8. Double beam: Free upper and botom beams. (a) Shear forces, (b) moment.

Figure 8 illustrates the moment and shear force diagrams of the double beam with a concentrated force at the middle upper beam. In this case, the shear force distribution between the upper and lower beams differs due to the placement of the force at the midpoint of the upper beam. The moment in the upper beam at the midpoint is significantly greater than that in the bottom beam.

IV. CONCLUSIONS

The paper successfully developed a finite element formulation for the double beam with a viscoelastic intermediate layer using classical beam theory. The displacements of the upper and lower beams were approximated using the Hermitian cubic interpolation functions to construct the beam stiffness matrix. The computed results were compared with the analytical solutions, demonstrating the reliability of the analytical approach in this study. The analysis of the beam displacements with different values of the layer stiffness parameter revealed a significant influence of this layer on the displacements of both the upper and lower beams in the double beam system.

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