An Improved Correction Technique for the Prediction of the Dynamic Response of a Beam under a Moving Vehicle

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ABSTRACT

This study presents a correction approach that can capture the discontinuities in the bending moment and shear force in the dynamic analysis of beam-like structures traveled by a moving vehicle. The proposed approach was based on the Dynamic Modal Acceleration Method (DyMAM) to correct the dynamic response of the supporting structure with a reduced number of vibration modes. The use of a two-axle vehicle model was adopted to consider the pitching effect in the presence of surface irregularity and damping. The interacting forces between the beam and vehicle were filtered to avoid undesirable high-frequency contributions. Subsequently, a new formulation for the entire vehicle-beam system was obtained. The corresponding equation was solved using the Newmark numerical scheme to obtain the system responses in each time step. A numerical example was illustrated, showing that the proposed method was in close agreement with previous correction solutions in the vehicle-beam interaction analysis.

Keywords-moving vehicle; vehicle-bridge interaction; dynamic modal acceleration method; modal equations of motion

I. INTRODUCTION

Vibration analysis of beam-like structures is an interesting aspect of structural design [1-2]. The study of the dynamic response of a beam induced by a moving load has long been an attractive topic [3-5]. The traditional Modal Displacement Method (MDM) is a widespread approach to determine the dynamic response of a beam and analyze Vehicle Beam Interaction (VBI) systems [6-10]. In MDM, the solution is obtained as a series expansion in terms of the eigenfunctions of the distributed system without damping and loading, having the advantage that the response quantities can be determined by the superposition of only a few first modes possessing low frequencies. Unfortunately, the exclusion of high-frequency modes leads to significant errors in the calculation of strains and stresses [11]. Once considering the bending moment and the shear force distribution of the beam, the MDM-based approach does not show the effectiveness of capturing the discontinuity and the jump of internal forces diagrams at the points where the vehicle axles are applied.

Several Modal Correction Methods (MCMs) have been proposed to involve the contribution of truncated higher-order modes to the structural response, such as the Mode Acceleration Method (MAM) [11-12], the Dynamic Correction Method (DCM) [13-14], and the Force Derivative Method (FDM) [15-16]. To the best of our knowledge, a limited amount of research has applied MCMs to VBI analysis. In [17-18], computational procedures were developed to analyze the dynamic response of a beam to single or multiple moving oscillators. The solution was the sum of modal series representation and quasistatic response. In [19-20], an approach based on DCM was introduced, aiming to consider the inertial, damping, and gravitational effects of moving masses and oscillators. In [21], a correction procedure was presented, in which the system response was separated into response components of low- and high-frequency contribution. These MCMs significantly improved the accuracy of structural dynamic response computations for the MDM when using a reduced set of eigenvectors. However, the effectiveness of MAM decreases in the case of high-frequency content loading. The solution provided in [20] seems to be quite mathematically cumbersome. In the procedure exploited in [21], the described correction method could fail when the loading function cannot be expressed in analytical form. Such problems have resulted in inconvenience in the computational process.

To overcome the limits of the above-mentioned MCMs, a MAM variant called the Dynamic Modal Acceleration Method (DyMAM) was presented, which can be used for both discrete and continuous structural systems [22]. The main idea of this approach was to introduce an additional dummy oscillator to filter each dynamic load on the structure to avoid undesirable high-frequency contributions. The output of this filter was used to correct the responses of the structure given by MDM. Due to the simple implementation and the fewer requirements in the excessive application, DyMAM can be operated in both deterministic and random dynamic loading cases [22]. As observed, no extension of this technique to predict the dynamic response of continuous structures traveled by a moving system has been carried out.

In this study, the DyMAM procedure was extended for dynamic response analysis of a beam-like structure subjected to a moving vehicle. It is possible to estimate the response of the supporting structure by using fewer eigenfunctions, avoiding a more cumbersome solving procedure. The vehicle was idealized as a rigid bar with two degrees of freedom and linked to the beam through two independent suspension units and tires set by linear mass-spring-damper systems. The interacting force caused by the vehicle axle on the beam was filtered through an elementary dynamic system. The equation of motion of the coupled VBI system based on a semi-analytical approach was established. Then, the dynamic response of the VBI system was obtained using the Newmark numerical scheme, which makes it easier to speed up the analysis. The advantage of the suggested method was shown by considering the quasistatic effect and capturing the stress discontinuities caused by the moving vehicle. Therefore, DyMAM provides a convenient procedure to improve accuracy in VBI analysis.

II. THEORETICAL MODELING OF THE VBI SYSTEM

A. Vehicle's Equations of Motion

Figure 1 shows the considered two-axle vehicle model. The vehicle body is simplified as a rigid bar and can move vertically by v_c and rotate φ_c around its center of mass. The terms m_c and J_c stand for mass and moment of inertia, respectively. Each wheel is represented by one vertical motion $v_{w,i}$, the corresponding mass $m_{w,i}$, and the tire characteristics including $k_{w,i}$ and $c_{w,i}$. The suspension unit is described by a linear spring dashpot with a stiffness coefficient $k_{s,I}$ and a damping coefficient $c_{s,i}$. The vehicle is assumed to run horizontally along the beam at a constant speed V, and L_i is the horizontal distance of the i-th axle to the center of the vehicle body.

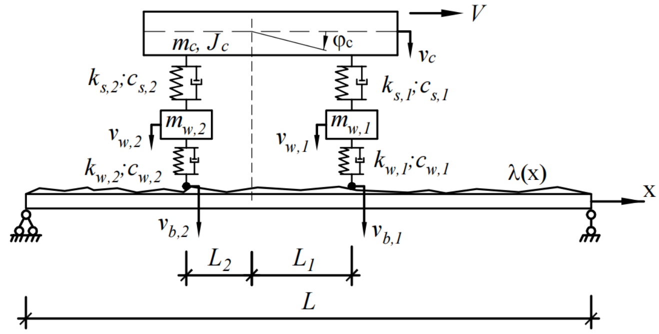

Fig. 1. The vehicle model moving along a beam.

Using D'Alembert's principle, the vehicle's equations of motion body are written as follows:

$$m_c\ddot{v}_c + c_{s,1}(\dot{v}_c - \dot{v}_{w,1} + \dot{\varphi}_c L_1) + k_{s,1}(v_c - v_{w,1} + \varphi_c L_1)$$
$$+c_{s,2}(\dot{v}_c - \dot{v}_{w,2} - \dot{\varphi}_c L_2) + k_{s,2}(v_c - v_{w,2} - \varphi_c L_2) = 0 \quad (1)$$

$$J_c\ddot{\varphi}_c + c_{s,1}L_1(\dot{v}_c - \dot{v}_{w,1} + \dot{\varphi}_c L_1) + k_{s,1}L_1(v_c - v_{w,1} + \varphi_c L_1)$$
$$-c_{s,2}L_2(\dot{v}_c - \dot{v}_{w,2} - \dot{\varphi}_c L_2) - k_{s,2}L_2(v_c - v_{w,2} - \varphi_c L_2) = 0 \quad (2)$$

The equations of motion of the wheels:

$$m_{w,1}\ddot{v}_{w,1} + c_{s,1}(\dot{v}_{w,1} - \dot{v}_c - \dot{\varphi}_c L_1) + k_{s,1}(v_{w,1} - v_c - \varphi_c L_1)$$
$$+c_{w,1}(\dot{v}_{w,1} - \dot{v}_{b,1}) + k_{w,1}(v_{w,1} - v_{b,1}) = 0 \quad (3)$$

$$m_{w,2}\ddot{v}_{w,2} + c_{s,2}(\dot{v}_{w,2} - \dot{v}_c + \dot{\varphi}_c L_2) + k_{s,2}(v_{w,2} - v_c + \varphi_c L_2)$$
$$+c_{w,2}(\dot{v}_{w,2} - \dot{v}_{b,2}) + k_{w,2}(v_{w,2} - v_{b,2}) = 0 \quad (4)$$

where the over-dot denotes the derivative quantities for time t, $v_{b,i}$ is the vertical displacement at the contact point between the i-th axle and the surface of the bridge. The interaction force $f_{c,I}$ caused by the i-th axle to the beam is given by:

$$f_{c,i} = c_{w,i}(\dot{v}_{w,i} - \dot{v}_{b,i}) + k_{w,i}(v_{w,i} - v_{b,i})$$
$$+m_{w,i}g + r_i m_c g \quad (5)$$

where r_i is the weight distribution coefficient of the vehicle body on the i-th wheel under a static load of vehicle self-weight $r_1 = \frac{L_2}{(L_1+L_2)}$ and $r_2 = \frac{L_1}{(L_1+L_2)}$, and g denotes the acceleration of gravity. The vertical displacement at the contact point $v_{b,I}$ is given by:

$$v_{b,i} = [w(x,t) + \lambda(x)]|_{x=x_i(t)} = \chi_i w(x_i(t),t) + \lambda(x_i(t)) \quad (6)$$

$$\dot{v}_{b,i} = \left.\frac{d[v_{b,i}(x,t)]}{dt}\right|_{x=x_i(t)} = \chi_i \frac{\partial w(x_i(t),t)}{\partial t} + \chi_i \frac{\partial w(x_i(t),t)}{\partial x}\dot{x}_i(t)$$
$$+\frac{d\lambda(x_i(t))}{dx}\dot{x}_i(t)$$
$$=\chi_i \dot{w}(x_i(t),t) + \chi_i w'(x_i(t),t)\dot{x}_i(t) + \lambda'(x_i(t))\dot{x}_i(t) \quad (7)$$

where the over prime means space derivative, $w(x_i(t),t)$ and $\lambda(x_i(t))$ are the vertical displacement and the surface roughness of the bridge corresponding to $x_i(t)$, that represents the location of the i-th axle in the structural coordinate system over time. χ_i is called the window function that describes the appearance of the axle of the vehicle on the bridge, which can be conveniently adopted by [20]:

$$\chi_i = \begin{cases} 1 \text{ for } 0 \le x_i(t) \le L, \\ 0 \text{ for } x_i(t) < 0 \text{ or } x_i(t) > L \end{cases} \quad (8)$$

B. The Equations of Motion of the Bridge using the Dynamic Modal Acceleration Method (DyMAM)

The partial differential equation of an elastic Euler-Bernoulli uniform beam subjected to a moving load can be expressed as:

$$\mu\frac{\partial^2 w(x,t)}{\partial t^2}+D(x,t)+EI\frac{\partial^4 w(x,t)}{\partial x^4}=\sum_{i=1}^{2}\chi_i f_{c,i}\delta[x-x_i(t)] \quad (9)$$

where EI, μ and $D(x,\ t)$ are flexural rigidity mass per unit of length and damping force per unit length of beam, respectively, and $\delta[]$ denotes the Dirac's delta function. Using the modal displacement method, the approximate solution of (9) can be represented as the superposition of the first N_m modal shapes of the beam:

$$w(x,t)=\sum_{n=1}^{N_m}W_n(x)T_n(t)=\ W^T(x)T(t)=w_{MDM}(x,t) \quad (10)$$

$$W(x)=\langle W_1(x)\quad\dots\quad W_n(x)\quad\dots\quad W_{N_m}(x)\rangle^T \quad (11)$$

$$T(t)=\langle T_1(t)\quad\dots\quad T_n(t)\quad\dots\quad T_{N_m}(t)\rangle^T \quad (12)$$

where $T_n(t)$ is the generalized modal coordinates, $W_n(x)$ is the normal vibration mode which is determined simultaneously with the n-th modal circular frequency ω_n as the solution of the eigenvalue problem:

$$\mu\omega_n^2W_n(x)=EI\frac{d^4}{dx^4}W_n(x) \quad (13)$$

$W_n(x)$ satisfies the boundary conditions and the orthogonality property:

$$\int_0^L\mu W_n(x)W_m(x)dx=\delta_{nm} \quad (14)$$

where δ_{nm} is the Kronecker delta. Substituting (10) into (9), then multiplying both parts of (9) by $W_m(x)$, and integrating into the range $[0,\ L]$, results in:

$$\ddot{T}_n(t)+2\xi_n\omega_n\dot{T}_n(t)+\omega_n^2T_n(t)=\sum_{i=1}^{2}\chi_i f_{c,i}W_n(x_i(t)) \quad (15)$$

where ξ_n is the damping ratio in the n-th mode of vibration, and $W_n(x_i(t))$ stands for the n-th normal vibration mode at $x_i(t)$. Equation (15) can be expressed in matrix form:

$$\ddot{T}+\kappa\dot{T}+\Omega^2\,T=W_L(x(t))Xf_c \quad (16)$$

where:

$$\kappa=diag[2\xi_n\omega_n] \quad (17)$$

$$\Omega=diag[\omega_n] \quad (18)$$

$$X=diag[\chi_i] \quad (19)$$

$$W_L\big(x(t)\big)=\begin{bmatrix}W_1\big(x_1(t)\big) & W_1\big(x_2(t)\big)\\ \dots & \dots\\ W_n\big(x_1(t)\big) & W_n\big(x_2(t)\big)\\ \dots & \dots\\ W_{N_m}\big(x_1(t)\big) & W_{N_m}\big(x_2(t)\big)\end{bmatrix} \quad (20)$$

$$\mathrm{f}_c=\begin{Bmatrix}f_{c,1}\\ f_{c,2}\end{Bmatrix} \quad (21)$$

where $diag[]$ defines the diagonal matrix. According to the DyMAM, the displacement of the beam can be expressed as [22]:

$$w(x,t)=w_{MDM}(x,t)+\Delta w_{DyMAM}(x,t)$$
$$=W^T(x)T(t)+Z^T(x)H(t)=w_{DyMAM}(x,t) \quad (22)$$

where $H(t)$ is the filter dynamic loading vector, and $Z(x)$ is the time-dependent vector collecting the residual flexibility matrix.

$$Z(x)=\big\langle Z\big(x,x_1(t)\big)\quad Z\big(x,x_2(t)\big)\big\rangle^T \quad (23)$$

$$H(t)=\langle H_1(t)\quad H_2(t)\rangle^T \quad (24)$$

where:

$$Z\Big(x,x_j(t)\Big)=G(x,x_j(t))\chi_j-\sum_{n=1}^{N_m}\frac{1}{\omega_n^2}W_n(x)W_n(x_j(t))\chi_j \quad (25)$$

where $G(x,\ x_j(t))$ presents the static Green's function of the beam that is the solution of (26) with appropriate boundary conditions.

$$EI\frac{d^4}{dx^4}G(x,x_j(t))=\delta\big[x-x_j(t)\big] \quad (26)$$

$H(t)$ is determined as the solution of the following equation:

$$\ddot{H}+\bar{\kappa}\dot{H}+\bar{\Omega}^2\,H=\bar{\Omega}^2\,Xf_c \quad (27)$$

where:

$$\bar{\kappa}=diag[2\bar{\xi}_1\bar{\omega}_1\quad 2\bar{\xi}_2\bar{\omega}_2] \quad (28)$$

$$\bar{\Omega}=diag[\bar{\omega}_1\quad \bar{\omega}_2] \quad (29)$$

where $\bar{\omega}_i$is the circular frequency of the filter associated with the dynamic load of the i-th axle, while $\bar{\xi}_i$ is the viscous damping ratio for the t-th filter, which is computed according to Rayleigh's damping.

$$\bar{\omega}_j=\begin{cases}\sqrt{\dfrac{EI\int_0^L\left[\frac{d^2}{dx^2}G\Big(x,x_j(t)\Big)\right]^2dx-\Phi_j^T\Omega^2\Phi_j}{\mu\int_0^L G^2\Big(x,x_j(t)\Big)dx-\Phi_j^T\Phi_j}} & \text{for } 0\le x_j(t)\le L,\\ 0 & \text{for } x_j(t)<0 \text{ or } x_j(t)>L\end{cases} \quad (30)$$

$$\bar{\xi}_j=\frac{1}{2\bar{\omega}_j}\theta_M+\frac{\bar{\omega}_j}{2}\theta_K \quad (31)$$

$$\Phi_j=\mu\int_0^L\boldsymbol{W}(x)G(x,x_j(t))dx \quad (32)$$

$$\theta_M=\frac{2}{\omega_{N_m}^2-\omega_1^2}\omega_1\omega_{N_m}\big(\omega_{N_m}\xi_1-\omega_1\xi_{N_m}\big) \quad (33)$$

$$\theta_K=\frac{2(\omega_{N_m}\xi_{N_m}-\omega_1\xi_1)}{\omega_{N_m}^2-\omega_1^2} \quad (34)$$

C. The Equations of Motion of the VBI System

Based on DyMAM, the vertical displacement of the bridge corresponding to $x_i(t)$ is expressed as follows:

$$w(x_i(t),t) \approx \sum_{n=1}^{N_m} W_n\big(x_i(t)\big)T_n(t) + \sum_{j=1}^{2} Z\big(x_i(t),x_j(t)\big)H_j(t) \quad (35)$$

The partial derivative of the displacement can be written as:

$$\dot{w}(x_i(t),t) \approx \sum_{n=1}^{N_m} W_n\big(x_i(t)\big)\dot{T}_n(t) + \sum_{j=1}^{2} Z_j\left(x_i(t),x_j(t)\right)\dot{H}_j(t) \quad (36)$$

$$w^{'}(x_i(t),t) \approx \sum_{n=1}^{N_m} W_n^{'}\big(x_i(t)\big)T_n(t) + \sum_{j=1}^{2} Z_j^{'}\left(x_i(t),x_j(t)\right)H_j(t) \quad (37)$$

$$\dot{x}_i(t) = \frac{dx_i(t)}{dt} = V \quad (38)$$

Substituting (36)-(38) into (6)-(7) results in:

$$v_{b,i} = \chi_i \sum_{n=1}^{N_m} W_n\big(x_i(t)\big)T_n(t) + \chi_i \sum_{j=1}^{2} Z\big(x_i(t),x_j(t)\big)H_j(t) + \lambda\big(x_i(t)\big) \quad (39)$$

$$\dot{v}_{b,i} = \chi_i \sum_{n=1}^{N_m} W_n\big(x_i(t)\big)\dot{T}_n(t) + \chi_i\dot{x}_i(t)\sum_{n=1}^{N_m} W_n^{'}\big(x_i(t)\big)T_n(t) + \dot{x}_i(t)\lambda'\big(x_i(t)\big) + \chi_i \sum_{j=1}^{2} Z\left(x_i(t),x_j(t)\right)\dot{H}_j(t) + \chi_i\dot{x}_i(t)\sum_{j=1}^{2} Z^{'}\big(x_i(t),x_j(t)\big)H_j(t) \quad (40)$$

Substituting (39)-(40) into (1)-(4) and aftere combining with (5) leads to the following equations of motion of the coupled VBI system in modal space, written symbolically as:

$$M\ddot{u} + C\dot{u} + Ku = P \quad (41)$$

where M, C, and K are the mass matrix, damping matrix, and stiffness matrix of the coupled vehicle-bridge system, respectively, $u, \dot{u}$ and $\ddot{u}$ denote the displacement, velocity, and acceleration vectors, respectively, and P denotes the force vector. To make the presentation clear in (41), the submatrices, subcolumn, and sub-row vectors will be denoted by quantities enclosed by [ ], { }, and ⟨ ⟩, respectively.

The matrices are defined as follows:

$$M = \begin{bmatrix} [I]_{N_m\times N_m} & [0] & \{0\} & \{0\} & [M_w]_{N_m\times 2} \\ [0] & [I]_{2\times 2} & \{0\} & \{0\} & [\bar{M}_w]_{2\times 2} \\ \langle 0\rangle & \langle 0\rangle & m_c & 0 & \langle 0\rangle \\ \langle 0\rangle & \langle 0\rangle & 0 & J_c & \langle 0\rangle \\ [0]_{2\times N_m} & [0]_{2\times 2} & \{0\} & \{0\} & diag\big[m_{w,i}\big]_{2\times 2} \end{bmatrix}$$

$$\mathrm{u} = \begin{Bmatrix} T(t) \\ H(t) \\ v_c \\ \varphi_c \\ \{v_{w,i}\} \end{Bmatrix} \quad (42)$$

where $[I]$ is the identity matrix.

$$[M_w] = \begin{bmatrix} m_{1,1} & m_{1,2} \\ \cdots & \cdots \\ m_{n,1} & m_{n,2} \\ \cdots & \cdots \\ m_{N_m,1} & m_{N_m,2} \end{bmatrix} \quad (43)$$

with $m_{n,i}=m_{w.i}W_n(x_i(t))\chi_i$.

$$[\bar{M}_w] = diag[\bar{m}_i], \text{ with } \bar{m}_i = m_{w,i}\bar{\omega}_i^2\chi_i \quad (44)$$

$$C = \begin{bmatrix} \kappa_{N_m\times N_m} & [0] & \{C_v\} & \{C_\varphi\} & [C_s]_{N_m\times 2} \\ [0] & \bar{\kappa}_{2\times 2} & \{\bar{C}_v\} & \{\bar{C}_\varphi\} & [\bar{C}_s]_{2\times 2} \\ \langle 0\rangle & \langle 0\rangle & \sum_{i=1}^{2} c_{s,i} & c_{s,1}L_1 - c_{s,2}L_2 & \langle C_{sv}\rangle \\ \langle 0\rangle & \langle 0\rangle & c_{s,1}L_1 - c_{s,2}L_2 & \sum_{i=1}^{2} c_{s,i}L_i^2 & \langle C_{s\varphi}\rangle \\ [C_w]_{2\times N_m} & [\bar{C}_w]_{2\times 2} & \langle C_{sv}\rangle^T & \langle C_{s\varphi}\rangle^T & diag\big[c_{s,i}+c_{w,i}\big]_{2\times 2} \end{bmatrix} \quad (45)$$

$$\{C_v\} = \langle c_1^v \quad \dots \quad c_n^v \quad \dots \quad c_{N_m}^v\rangle^T \quad (46)$$

with $c_n^v = -\sum_{i=1}^{2} c_{s,i} W_n(x_i(t))\chi_i$.

$$\{C_\varphi\} = \langle c_1^\varphi \quad \dots \quad c_n^\varphi \quad \dots \quad c_{N_m}^\varphi\rangle^T \quad (47)$$

with $c_n^\varphi = -c_{s,1}L_1W_n(x_1(t))\chi_1 + c_{s,2}L_2W_n(x_2(t))\chi_2$.

$$[C_s] = \begin{bmatrix} c_{1,1}^s & c_{1,2}^s \\ \cdots & \cdots \\ c_{n,1}^s & c_{n,2}^s \\ \cdots & \cdots \\ c_{N_m,1}^s & c_{N_m,2}^s \end{bmatrix} \quad (48)$$

with $c_{n,i}^s = c_{s,i}W_n(x_i(t))\chi_i$.

$$\{\bar{C}_v\} = \langle \bar{c}_1^v \quad \bar{c}_2^v\rangle^T \text{ with } \bar{c}_i^v = -c_{s,i}\bar{\omega}_i^2\chi_i \quad (49)$$

$$\{\bar{C}_\varphi\} = \langle -\bar{c}_1^\varphi \quad \bar{c}_2^\varphi\rangle^T \text{ with } \bar{c}_i^\varphi = c\bar{l}_i^2{}_{i_{s,i}} \quad (50)$$

$$[\bar{C}_s] = diag[\bar{c}_i^s] \text{ with } \bar{c}_i^s = c_{s,i}\bar{\omega}_i^2\chi_i \quad (51)$$

$$\langle C_{sv}\rangle = \langle -c_{s,1} \quad -c_{s,2}\rangle \quad (52)$$

$$\langle C_{s\varphi}\rangle = \langle -c_{s,1}L_1 \quad c_{s,2}L_2\rangle \quad (53)$$

$$[C_w] = \begin{bmatrix} c_{1,1}^w & \cdots & c_{1,n}^w & \cdots & c_{1,N_m}^w \\ c_{2,1}^w & \cdots & c_{2,n}^w & \cdots & c_{2,N_m}^w \end{bmatrix} \quad (54)$$

with $c_{i,n}^w = -c_{w,i}W_n(x_i(t))\chi_i$.

$$[\bar{C}_w] = \begin{bmatrix} \bar{c}_{1,1}^w & \bar{c}_{1,2}^w \\ \bar{c}_{2,1}^w & \bar{c}_{2,2}^w \end{bmatrix} \quad (55)$$

with $c_{i,j}^w = -c_{w,i}Z\big(x_i(t),x_j(t)\big)\chi_i$.

$$K = \begin{bmatrix} \Omega_{N_m\times N_m}^2 & [0] & \{K_v\} & \{K_\varphi\} & [K_s]_{N_m\times 2} \\ [0] & \bar{\Omega}_{2\times 2}^2 & \{\bar{K}_v\} & \{\bar{K}_\varphi\} & [\bar{K}_s]_{2\times 2} \\ \langle 0\rangle & \langle 0\rangle & \sum_{i=1}^{2} k_{s,i} & k_{s,1}L_1 - k_{s,2}L_2 & \langle K_{sv}\rangle \\ \langle 0\rangle & \langle 0\rangle & k_{s,1}L_1 - k_{s,2}L_2 & \sum_{i=1}^{2} k_{s,i}L_i^2 & \langle K_{s\varphi}\rangle \\ [K_w]_{2\times N_m} & [\bar{K}_w]_{2\times 2} & \langle K_{sv}\rangle^T & \langle K_{s\varphi}\rangle^T & diag\big[k_{s,i}+k_{w,i}\big]_{2\times 2} \end{bmatrix} \quad (56)$$

$$[K_w] = \begin{bmatrix} k_{1,1}^w & \dots & k_{1,n}^w & \dots & k_{1,N_m}^w \\ k_{2,1}^w & \dots & k_{2,n}^w & \dots & k_{2,N_m}^w \end{bmatrix} \quad (57)$$

with $k_{i,n}^w = -k_{w,i}W_n(x_i(t))\chi_i - c_{w,i}\dot{x}_i(t)W_n^{'}(x_i(t))\chi_i$.

$$[\bar{K}_w] = \begin{bmatrix} \bar{k}_{1,1}^w & \bar{k}_{1,2}^w \\ \bar{k}_{2,1}^w & \bar{k}_{2,2}^w \end{bmatrix} \quad (58)$$

with:

$$\bar{k}_{i,j}^w = -k_{w,i} Z\big(x_i(t), x_j(t)\big)\chi_i - c_{w,i}\dot{x}_i(t) Z'\big(x_i(t), x_j(t)\big)\chi_i$$

The submatrices $[K]$ have a similar form as the submatrices $[C]$ and can be obtained by replacing the damping coefficient with the stiffness coefficient.

$$P = \begin{Bmatrix} \{F\} \\ \{\bar{F}_a\} \\ 0 \\ 0 \\ \{\bar{F}_w\} \end{Bmatrix}, \qquad \{F\} = \begin{Bmatrix} f_1 \\ \dots \\ f_n \\ \dots \\ f_{N_m} \end{Bmatrix} \quad (59)$$

with $f_n = \sum_{i=1}^{2} (m_{w,i} + r_i m_c) g W_n\big(x_i(t)\big)\chi_i$ (60)

$$\{\bar{F}_a\} = \begin{Bmatrix} \bar{f}_{a,1} \\ \bar{f}_{a,2} \end{Bmatrix} \text{ with } \bar{f}_{a,i} = (m_{w,i} + r_i m_c) g \bar{\omega}_i^2 \chi_i \quad (61)$$

$$\{\bar{F}_w\} = \begin{Bmatrix} \bar{f}_{w,1} \\ \bar{f}_{w,2} \end{Bmatrix} \quad (62)$$

with $\bar{f}_{w,i} = c_{w,i}\dot{x}_i(t)\lambda'\big(x_i(t)\big) + k_{w,i}\lambda\big(x_i(t)\big)$.

Since the positions of excitation caused by the wheels depend on the vehicle velocity and the time increment, (41) must be modified and updated after each time step. It can be solved by direct time integration to obtain the dynamic responses of the system simultaneously.

III. EVALUATION OF BENDING MOMENT AND SHEAR FORCE

Once (41) is solved, the problem of calculating the bending moment and shear force of the beam can be evaluated by differentiating (10) or (22) for the spatial coordinate x. In the case of homogeneous and uniform Euler-Bernoulli beam, the bending moment and shear force are given by:

$$M(x,t) = -EIw_b''(x,t) \quad (63)$$

$$Q(x,t) = -EIw_b'''(x,t) \quad (64)$$

Using (10), the bending moment and shear force based on MDM are given by:

$$M_{MDM}(x,t) = -EIW''(x)^T T(t) \quad (65)$$

$$Q_{MDM}(x,t) = -EIW'''(x)^T T(t) \quad (66)$$

Using (22) and (25), the bending moment and shear force are given by:

$$M_{DyMAM}(x,t) = -EIW''(x)^T T(t)$$
$$-EI[G''(x,x(t))^T - W''(x)^T \Omega^{-2} W_L(x(t))]XH(t) \quad (67)$$

$$Q_{DyMAM}(x,t) = -EIW'''(x)^T T(t)$$
$$-EI[G'''(x,x(t))^T - W'''(x)^T \Omega^{-2} W_L(x(t))]XH(t) \quad (68)$$

where $G''\big(x,x(t)\big)$ and $G'''\big(x,x(t)\big)$ are the vector of order 2 listing the partial derivative of the static Green's function $G\big(x,x_i(t)\big)$, and $W''(x), W'''(x)$ are the vectors of order N_m listing the normal vibration modes $W_n(x)$ with respect to x.

IV. NUMERICAL EXAMPLE

The applicability of DyMAM in the VBI analysis validation was conducted based on numerical examples. The proposed algorithms were implemented in the MATLAB environment [23]. Without loss of generality, let us consider a single span simply supported beam. The natural circular frequencies of the beam are calculated by the solution of the eigenvalue for free vibration is:

$$\omega_n = \left(\frac{n\pi}{L}\right)^2 \sqrt{\frac{EI}{\mu}}$$

and the modal shape [24]:

$$W_{b,n}(x) = \sqrt{\frac{2}{\mu L}} \sin\left(\frac{n\pi x}{L}\right).$$

According to (5.8) in [24], the static Green's function of the simply supported beam is given by:

$$G(x,x_i(t)) = \begin{cases} \frac{1}{6EIL}[-(L-x_i(t))x^2 + (2L^2 - 3Lx_i(t) + x_i^2(t))x_i(t)]x & \text{for } x \le x_i(t) \\ \frac{1}{6EIL}[-(L-x)x_i^2(t) + (2L^2 - 3Lx + x^2)x]x_i(t) & \text{for } x \ge x_i(t) \end{cases} \quad (69)$$

The following mechanical parameters were chosen for the beam and the vehicle: span length L=40 m, mass per unit length $\mu=12\times10^3$ kg/m, flexural rigidity $EI=12.75\times10^{10}$ Nm^2, and the damping ratios of the beam were set to be ξ=0.02, vehicle body mass m_c=36000 kg, mass moment of inertia $J_c=1.44\times10^5$ kgm^2, axle mass $m_{w,1}=m_{w,2}$=2000 kg, suspension stiffness $k_{s,1}=k_{s,2}=9\times10^6$ N/m, suspension damping $c_{s,1}=c_{s,2}=72\times10^3$ Ns/m, tire stiffness $k_{w,1}=k_{w,2}=36\times10^6$ N/m, $c_{w,1}=c_{w,2}=72\times10^3$ Ns/m, axle distance $L_1=L2$=1 m, and velocity V=10 m/s. It was assumed that the front axle of the vehicle enters the beam at the instant t=0 s.

The surface irregularity was assumed in the form of a harmonic function represented by:

$$\lambda(x) = \left(\frac{d_r}{2}\right)\left[1 - \cos\left(\frac{2\pi x}{l_r}\right)\right] \quad (70)$$

where d_r and l_r are the surface irregularity depth and length, respectively. The value of two ratios was adopted as:

$$\frac{48EId_r}{(m_c + \sum m_i) g L^3} = 0.05, \qquad \frac{L}{l_r = 10}.$$

Equation (41) was solved using the Newmark-β method, with the coefficients β=1/6 and γ=1/2. The time step was chosen as Δt=0.0001 s. For comparison, the numerical analytical result of the DyMAM procedure was compared with the results evaluated by MDM and MAM [20]. The first five modes of vibration were retained in the analysis and the Newmark-β integral method was used to find dynamic responses in all approaches. Figures 2 and 3 show the time histories of the vertical displacement of the vehicle body and the displacement at the mid-span of the beam based on the considered approaches. Furthermore, Figures 4 and 5 show the bending moment and the shear force distribution at t=2 s,. Overall, a good agreement was obtained from the results.

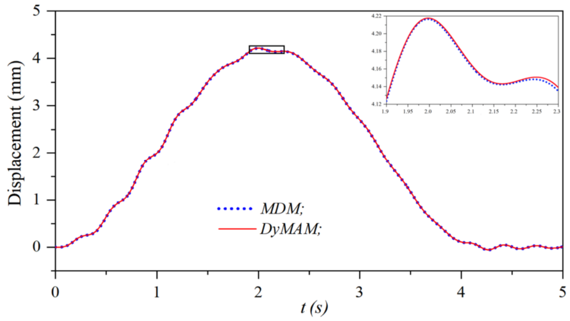

Fig. 2. Vertical displacement of vehicle body versus time.

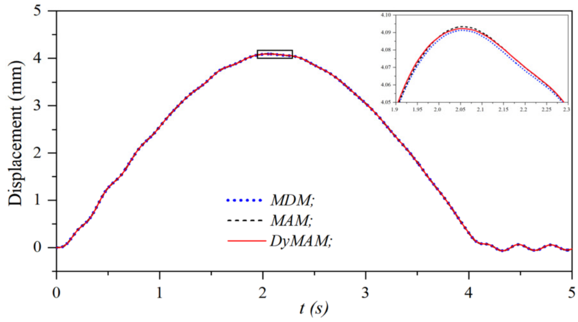

Fig. 3. Vertical displacement at mid-span of the beam versus time.

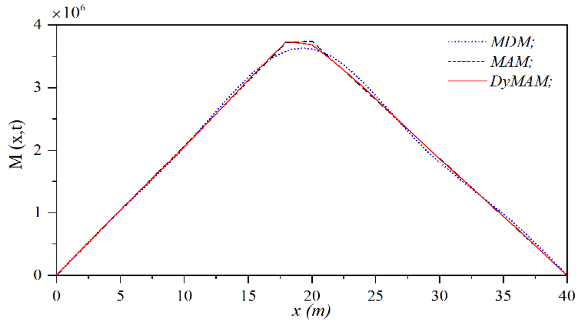

Fig. 4. Bending moment distributions along the beam at the instant t=2 s.

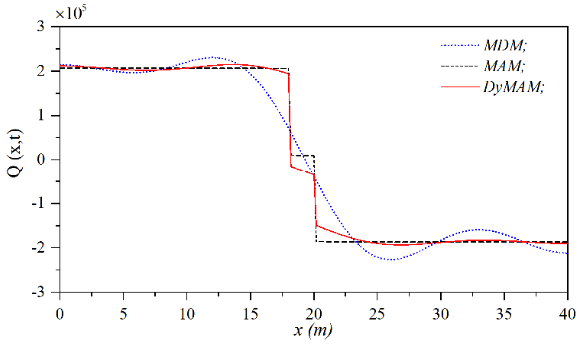

Fig. 5. Shear force distribution along the beam at the instant t=2 s.

Figure 2 illustrates that the solid line coincides with the dotted line, which means that the vertical displacement of the vehicle body determined by the DyMAM approach was the same as the one given by the MDM solution. Figure 3 shows the displacement in the mid-span of the beam. The solid line was very close to the dashed line, but there was a slight deviation between them. The cause of this was that the response of the beam in MAM was evaluated as the sum of the MDM and the quasi-static response, which considers the quasi-static displacements of the beam under the quasi-static interaction forces transmitted by the moving vehicle associated with the truncated higher-order eigenfunctions [20]. DyMAM agrees with both MDM and MAM in calculating the dynamic response of the beam. The shortcoming of the two methods is that MDM does not consider the influence of the quasistatic effect and MAM neglects the inertial and damping effects of the vehicle.

Figure 4 shows the bending moment distribution along the beam calculated at t=2 s. It can be seen that the DyMAM approach converged with the MAM. Figure 5 shows the shear force distributions of the beam at t=2 s. It can be easily seen that the difference in the convergence of the two correction methods is quite considerable. In particular, the differences at the two jump locations and the segment between them are the most pronounced. With a speed of 10 m/s, at t=2 s, the axles move 20 m, which means that the front axle of the vehicle is in the middle of the beam while the rear axle is 2m to the left.

As shown in Figures 4 and 5, at positions x_1=20 m and x_2=18 m, MDM was unable to capture any discontinuities in the bending moment and shear force. Meanwhile, there were two points of discontinuity captured by the MAM and the DyMAM approaches with different levels of accuracy. Two points of discontinuity appear since the two-axle vehicle model was employed and the whole vehicle stood on the beam at that time. On the other hand, the jumps in the shear force can be clearly observed when applying MAM and DyMAM. As shown in Figure 4, the jumps occur at two points and divide the shear force distribution into three segments. In the case of MAM, there was no change in each of these segments. However, in the case of DyMAM, each segment showed a change in the nonlinear form. This phenomenon can be explained by the dynamic effect of the vehicle: MAM only considers gravitational effects and excludes damping and inertial effects induced by the motion of the moving vehicle. Meanwhile, the inertial, damping, and gravitational effects resulting from the moving vehicle are included in DyMAM. Furthermore, the pitching effect also contributes to the dynamic response of the beam, further complicating things.

V. CONCLUSIONS

This study formulated the motion equations of a beam subjected to a moving vehicle in matrix form, based on the dynamic modal acceleration method, and validated them through a numerical example. The quasistatic effect of the beam transmitted by the moving vehicle associated with the contribution of high modes was considered. The DyMAM correction was used for the first time in VBI analysis. This approach has some advantageous features compared to the previous modal correction methods, as follows:

- The dynamic response of the beam and vehicle can be calculated directly by using a step-by-step integration procedure, as it supplies an adequate approximation in calculating the response of the beam in VBI analysis.

- The method can consider both actions of gravitational, damping, and inertial effects due to the moving vehicle.
- The performances of DyMAM can capture the discontinuities in the bending moment and shear force and the jump in the shear force.

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REFERENCES

[1] H. T. Duy, N. D. Diem, G. V. Tan, V. V. Hiep, and N. V. Thuan, "Stochastic Higher-order Finite Element Model for the Free Vibration of a Continuous Beam resting on Elastic Support with Uncertain Elastic Modulus," *Engineering, Technology & Applied Science Research*, vol. 13, no. 1, pp. 9985–9990, Feb. 2023, https://doi.org/10.48084/etasr.5456.

[2] P. C. Nguyen, "Nonlinear Inelastic Earthquake Analysis of 2D Steel Frames," *Engineering, Technology & Applied Science Research*, vol. 10, no. 6, pp. 6393–6398, Dec. 2020, https://doi.org/10.48084/etasr.3855.

[3] T. D. Hien, N. D. Hung, N. T. Hiep, G. V. Tan, and N. V. Thuan, "Finite Element Analysis of a Continuous Sandwich Beam resting on Elastic Support and Subjected to Two Degree of Freedom Sprung Vehicles," *Engineering, Technology & Applied Science Research*, vol. 13, no. 2, pp. 10310–10315, Apr. 2023, https://doi.org/10.48084/etasr.5464.

[4] T. D. Hien, N. D. Hung, N. T. Kien, and H. C. Noh, "The variability of dynamic responses of beams resting on elastic foundation subjected to vehicle with random system parameters," *Applied Mathematical Modelling*, vol. 67, pp. 676–687, Mar. 2019, https://doi.org/10.1016/j.apm.2018.11.018.

[5] J. P. Yang, "Theoretical Formulation of Amplifier–Vehicle–Bridge System Based on Sophisticated Vehicle Model," *Journal of Vibration Engineering & Technologies*, vol. 10, no. 2, pp. 789–794, Feb. 2022, https://doi.org/10.1007/s42417-021-00409-4.

[6] L. Ding, H. Hao, and X. Zhu, "Evaluation of dynamic vehicle axle loads on bridges with different surface conditions," *Journal of Sound and Vibration*, vol. 323, no. 3, pp. 826–848, Jun. 2009, https://doi.org/10.1016/j.jsv.2009.01.051.

[7] E. Esmailzadeh and N. Jalili, "Vehicle–passenger–structure interaction of uniform bridges traversed by moving vehicles," *Journal of Sound and Vibration*, vol. 260, no. 4, pp. 611–635, Feb. 2003, https://doi.org/10.1016/S0022-460X(02)00960-4.

[8] P. Lou, "Vertical dynamic responses of a simply supported bridge subjected to a moving train with two-wheelset vehicles using modal analysis method," *International Journal for Numerical Methods in Engineering*, vol. 64, no. 9, pp. 1207–1235, 2005, https://doi.org/10.1002/nme.1426.

[9] S. S. Law and X. Q. Zhu, "Bridge dynamic responses due to road surface roughness and braking of vehicle," *Journal of Sound and Vibration*, vol. 282, no. 3, pp. 805–830, Apr. 2005, https://doi.org/10.1016/j.jsv.2004.03.032.

[10] J. P. Yang, "Theoretical Formulation of Three-Mass Vehicle Model for Vehicle–Bridge Interaction," *International Journal of Structural Stability and Dynamics*, vol. 21, no. 07, Jul. 2021, Art. no. 2171004, https://doi.org/10.1142/S0219455421710048.

[11] H. L. Soriano and F. V. Filho, "On the modal acceleration method in structural dynamics. Mode truncation and static correction," *Computers & Structures*, vol. 29, no. 5, pp. 777–782, Jan. 1988, https://doi.org/10.1016/0045-7949(88)90345-8.

[12] P. Leger and E. L. Wilson, "Modal summation methods for structural dynamic computations," *Earthquake Engineering & Structural Dynamics*, vol. 16, no. 1, pp. 23–27, 1988, https://doi.org/10.1002/eqe.4290160103.

[13] G. Borino and G. Muscolino, "Mode-superposition methods in dynamic analysis of classically and non-classically damped linear systems," *Earthquake Engineering & Structural Dynamics*, vol. 14, no. 5, pp. 705–717, 1986, https://doi.org/10.1002/eqe.4290140503.

[14] A. D'Aveni and G. Muscolino, "Improved dynamic correction method in seismic analysis of both classically and non-classically damped structures," *Earthquake Engineering & Structural Dynamics*, vol. 30, no. 4, pp. 501–517, 2001, https://doi.org/10.1002/eqe.20.

[15] C. J. Camarda, R. T. Haftka, and M. F. Riley, "An evaluation of higher-order modal methods for calculating transient structural response," *Computers & Structures*, vol. 27, no. 1, pp. 89–101, Jan. 1987, https://doi.org/10.1016/0045-7949(87)90184-2.

[16] M. A. Akgün, "A New Family Of Mode-Superposition Methods For Response Calculations," *Journal of Sound and Vibration*, vol. 167, no. 2, pp. 289–302, Oct. 1993, https://doi.org/10.1006/jsvi.1993.1336.

[17] A. V. Pesterev and L. A. Bergman, "An Improved Series Expansion of the Solution to the Moving Oscillator Problem," *Journal of Vibration and Acoustics*, vol. 122, no. 1, pp. 54–61, Jan. 1999, https://doi.org/10.1115/1.568436.

[18] A. V. Pesterev, B. Yang, L. A. Bergman, and C.-A. Tan, "Response of Elastic Continuum Carrying Multiple Moving Oscillators," *Journal of Engineering Mechanics*, vol. 127, no. 3, pp. 260–265, Mar. 2001, https://doi.org/10.1061/(ASCE)0733-9399(2001)127:3(260).

[19] B. Biondi, G. Muscolino, and A. Sidoti, "Methods for Calculating Bending Moment and Shear Force in the Moving Mass Problem," *Journal of Vibration and Acoustics*, vol. 126, no. 4, pp. 542–552, Dec. 2004, https://doi.org/10.1115/1.1804992.

[20] B. Biondi and G. Muscolino, "New improved series expansion for solving the moving oscillator problem," *Journal of Sound and Vibration*, vol. 281, no. 1, pp. 99–117, Mar. 2005, https://doi.org/10.1016/j.jsv.2004.01.018.

[21] C. Bilello, M. Di Paola, and S. Salamone, "A correction method for the analysis of continuous linear one-dimensional systems under moving loads," *Journal of Sound and Vibration*, vol. 315, no. 1, pp. 226–238, Aug. 2008, https://doi.org/10.1016/j.jsv.2008.01.040.

[22] A. Palmeri and M. Lombardo, "A new modal correction method for linear structures subjected to deterministic and random loadings," *Computers & Structures*, vol. 89, no. 11, pp. 844–854, Jun. 2011, https://doi.org/10.1016/j.compstruc.2011.02.020.

[23] "MathWorks." [Online]. Available: https://www.mathworks.com/.

[24] L. Frýba, *Vibration of Solids and Structures Under Moving Loads*. Prague, Czech Republic: Thomas Telford, 1999.