Investigation of the Eigenvector of Stochastic Finite Element Methods of Functionally Graded Beams with Random Elastic Modulus

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Received: 26 April 2023 | Revised: 5 June 2023 | Accepted: 6 June 2023

 DOI: https://doi.org/10.48084/etasr.5991

ABSTRACT

This paper presents a stochastic finite element method to calculate the variation of eigenvalues and eigenvectors of functionally graded beams. The modulus of functionally graded material is assumed to have spatial uncertainty as a one-dimensional random field. The formulation of the stochastic finite element method for the functionally graded beam due to the randomness of the elastic modulus of the beam is given using the first-order perturbation approach. This approach was validated with Monte Carlo simulation in previous studies using spectral representation to generate the random field. The statistics of the beam responses were investigated using the first-order perturbation method for different fluctuations of the elastic modulus. A comparison of the results of the stochastic finite element method with the first-order perturbation approach and the Monte Carlo simulation showed a minimal difference.

Keywords-perturbation method; eigenvector; FGM beam; FEM

I. INTRODUCTION

Functionally Graded Materials (FGMs) have attracted much attention the recent years [1-8]. The idea of FGM was introduced in 1984 when the space aircraft project was underway in Japan [9]. Compared to generally isotropic homogeneous materials, the FGM beams under consideration can acquire a high degree of uncertainty in their material constants. FGMs are considered one of the most promising candidates for advanced composites in various industries, including aerospace, automotive, electronics, and, most recently, biomedical and aviation [10-11]. Structures constructed of FGM have attracted attention in recent years. In [12], the free vibration property of a functionally graded beam was used with axial or transverse material gradation by thickness according to the power law, using the Finite Element Method (FEM). In [13], an innovative method was presented to create an FGM beam finite element by deriving approximation functions from the exact general solution to the static section of the governing equations. In [14], the geometric nonlinearity of Von Karman was considered in an Euler-Bernoulli beam to study the nonlinear vibration of beams constructed of functionally graded materials.

Many researchers have studied problems in mechanics with random parameters [15-18]. With stochastic problems of continuous systems, analytical methods are almost always challenging to find solutions, often using numerical methods such as stochastic finite element analysis [19-22] and stochastic isogeometric analysis [4, 23]. In [24], the variation of vehicle vibration frequency was studied with random stiffness and mass parameters of the vehicle. In [25], stochastic dynamic problems of a functionally graded material with random material properties were studied. In [26], a stochastic finite element with uncertain Poisson's ratio was presented for bending plates. Stochastic FEM using the weighted integration approach for the static problem of nonuniform columns was presented in [27]. In [28], the Karhunen-Loève expansion was used to develop a stochastic spectral isogeometric analysis for the linear elasticity problem. In [29], the first-order reliability method and Monte Carlo Simulation (MCS) were used to calculate the reliability of laminated composite shells. However, very few studies used the perturbation technique to determine eigenvector problems. This study attempted to extend [30-31], using the results obtained on the eigenvalues as a basis to develop the eigenvector problems of FGM beams as the form of the perturbation-based stochastic FEM.

II. FINITE ELEMENT FORMULATION

At first, the finite element formulation for the functionally graded beam was formulated with the case of deterministic system parameters. Figure 1 illustrates the functionally graded beam with length L in the mid-surface coordinate system.

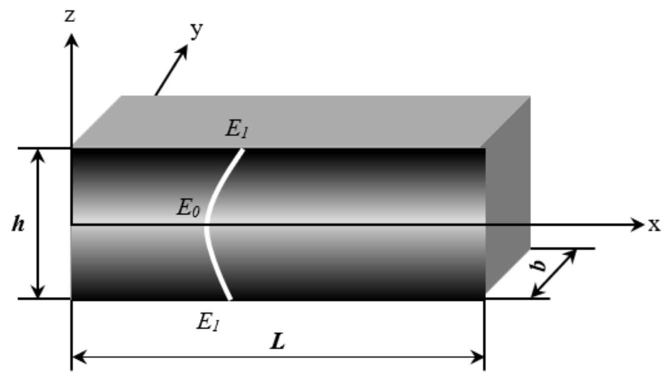

Fig. 1. Coordinates and geometry of the functionally graded beam.

Following the feature of the FGM, Young's modulus and mass density were assumed to vary continuously along the thickness direction (*z*-axis) according to the exponential function. It was also assumed that the elastic modulus and mass density was symmetric to the mid-surface, as given in the following expression:

$$E(z) = E_0 e^{\beta|z|}; \quad \rho(z) = \rho_0 e^{\beta|z|} \tag{1}$$

where E_0 and ρ_0 are the elasticity modulus and mass density in the mid-plane ($z=0$), respectively, and β defines the variation trend along the z-axis. A 2-node finite element beam was adopted, having Hermite functions as shape functions. The respective nodes have two degrees of freedom: translation w and rotation θ. Accordingly, the vertical displacement along the z-direction w can be interpolated as follows:

$$w(x) = [N_1 \quad N_2 \quad N_3 \quad N_4]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \tag{2}$$

where N_i are the interpolation functions:

$$\begin{aligned} N_1 &= 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} \\ N_2 &= x - \frac{2x^2}{L} + \frac{x^3}{L^2} \\ N_3 &= \frac{3x^2}{L^2} - \frac{2x^3}{L^3} \\ N_4 &= -\frac{x^2}{L} + \frac{x^3}{L^2} \end{aligned} \tag{3}$$

The strain energy of the beam element can be written as:

$$U_e = \int_0^l \frac{1}{2} b(x) D w_{,xx}^2 \, dx \tag{4}$$

Furthermore, the kinetic energy of the beam element can be written as follows:

$$\begin{aligned} T_e &= \int_0^l \frac{1}{2} \int_{-h/2}^{h/2} b(x)\rho(z) dz dx \, \dot{w}^2 \\ &= \int_0^l \frac{1}{2} m(x) \dot{w}^2 dx \end{aligned} \tag{5}$$

where, $m(x) = b(x) \int_{-h/2}^{h/2} \rho(z) dz$ and $D = \int_{-h/2}^{h/2} E(z) z^2 \, dz$.

Hamilton's principle was used to find the eigenequation for the FGM beam structures within the FEM framework. Before that, it is necessary to have the strain and kinetic energy in terms of the nodal displacement vector. The final equation of motion can be derived by applying the Hamilton's principle:

$$\delta \int_{t_1}^{t_2} \left(\sum_{e=1}^{N} U_e - \sum_{e=1}^{N} T_e \right) dt = 0 \tag{6}$$

where N denotes the number of finite elements used in the FGM beam model. Taking the variation concerning harmonic motion, the following equation emerges for the eigenvalues and vectors:

$$(K - \lambda M) q = 0 \tag{7}$$

The frequency equation that provides the solution for frequencies is obtained as:

$$|K - \lambda M| = 0 \tag{8}$$

In these equations, K and M denote the assembled global stiffness and mass of the FGM beam, respectively, and λ_i denotes the square of the circular frequency ω_i.

III. PROBABILISTIC ANALYSIS

The perturbation method is the most commonly used technique for studying uncertain systems. This method uses a Taylor series to construct an analytical solution for the variance of the desired response quantities by extending each random variable in an uncertain system around its associated mean value, such as the eigenvalues and eigenvectors of a structure, due to the smallest variations in the random variables. The stiffness and mass matrices of the beam and plate and their responses are represented in terms of a Taylor series formulation concerning the parameters centered at the mean values. The Taylor series is often only extended to the first-order perturbation method. The elastic modulus is described as a one-dimensional Gaussian homogeneous random field with zero mean $R(x)$, given by:

$$E(x, z) = E_0[1 + R(x)] e^{\beta|z|} \tag{9}$$

The random field of elastic modulus in the element is approximated by averaging random variables within it:

$$E \approx E_0 \left[1 + \frac{r_1 + r_2 + \ldots + r_n}{nr} \right] \tag{10}$$

Let $r = \langle r_k \rangle,\ k = 1, nr$ be a vector of the random variable, where nr is the number of coefficients of the random variable vector r. Since two kinds of random parameters have to be considered, elastic modulus and mass density, the vector r is represented as $r = \langle s, t \rangle$, where s and t are the random variable vectors for elastic modulus and mass density, respectively. For the mid-point rule, $= \langle s_1, s_2, \ldots, s_{nr_s} \rangle$, $t = \langle t_1, t_2, \ldots, t_{nr_s} \rangle$, and $nr_s, nr_t = ne$ is the number of finite elements in the domain. That is, each random variable is associated with the corresponding finite element. In the case of the local average scheme, $s = \langle s_1, s_2, \ldots, s_{nr_s} \rangle$, $nr_s = ne \times np$, where np is the number of integration points within each finite element. Similarly, $t = \langle t_1, t_2, \ldots, t_{nr_s} \rangle$, $nr_t = ne \times np$. Here it is assumed that the mean of the respective random variable is zero: $E[s] = E[t] = 0$.

The Taylor series of system matrices and vectors is:

$$K(r) = K_0 + K_{,i}\Delta s_i + K_{,i}\Delta t_i + \frac{1}{2}\left(K_{,ij}\Delta s_i\Delta s_j + K_{,ij}\Delta t_i\Delta t_j + 2K_{,ij}\Delta s_i\Delta t_j\right) \quad (11)$$

$$M(r) = M_0 + M_{,i}\Delta s_i + M_{,i}\Delta t_i + \frac{1}{2}\left(M_{,ij}\Delta s_i\Delta s_j + M_{,ij}\Delta t_i\Delta t_j + 2M_{,ij}\Delta s_i\Delta t_j\right) \quad (12)$$

$$X(r) = X_0 + X_{,i}\Delta s_i + X_{,i}\Delta t_i + \frac{1}{2}\left(X_{,ij}\Delta s_i\Delta s_j + X_{,ij}\Delta t_i\Delta t_j + 2X_{,ij}\Delta s_i\Delta t_j\right) \quad (13)$$

$$\lambda(r) = \lambda_0 + \lambda_{,i}\Delta s_i + \lambda_{,i}\Delta t_i + \frac{1}{2}\left(\lambda_{,ij}\Delta s_i\Delta s_j + \lambda_{,ij}\Delta t_i\Delta t_j + 2\lambda_{,ij}\Delta s_i\Delta t_j\right) \quad (14)$$

Here, $r = \langle s, t\rangle$

$$P_0 = P|_{r=0}, P_{,i} = \left.\frac{\partial P}{\partial r_i}\right|_{r=0},$$

$$P_{,ij} = \left.\frac{\partial P}{\partial r_i \partial r_j}\right|_{r=0}, \ i,j = 1,2,\ldots,nr \quad (15)$$

$$\Delta s_i = s_i - \bar{s}_i = s_i, \Delta t_i = t_i - \bar{t}_i = t_i \quad (16)$$

It is apparent that stiffness is only a function of random elastic modulus and the mass matrix is of random mass density. The summation is implied with the repeated indices. By substituting the expanded expressions for the system matrices into the eigenanalysis expression, the equation in the same order can be obtained as follows:

The 0-th order equation:

$$(K_0 - \lambda_0^k M_0)X_0^k = 0 \quad (17)$$

The 1-st order equation:

$$(K_0 - \lambda_0^k M_0)X_0^k = -(K_{,i} - \lambda_0^k M_{,i} - \lambda_{,i}^k M_0)X_0^k \quad (18)$$

Considering only the linear terms, the linear variance of the eigenvalue and eigenvector is:

$$\lambda_{,i}^k = X^{kT}\left(K_{,i}^{(s)} - \lambda^k\, M_{,i}^{(t)}\right)X^k \quad (19)$$

$$Var_{(L)}(X) = E[(X - \bar{X}_L)(X - \bar{X}_L)^T]$$
$$= E\left[\left(X_{,i}^{(s)}s_i + X_{,i}^{(t)}t_i\right)\left(X_{,j}^{(s)T}s_j + X_{,j}^{(t)T}t_j\right)\right]$$
$$= X_{,i}^{(s)}X_{,j}^{(s)T}E\left[s_is_j\right] + X_{,i}^{(t)}t_iX_{,j}^{(t)T}t_iE\left[t_it_j\right]$$
$$+2X_{,i}^{(s)}s_iX_{,j}^{(t)T}E\left[s_it_j\right] \quad (20)$$

where:

$$X_{,i}^{(s)} = -S_o^{-1}(K_{,i}^{(s)} - \lambda_0 M_{,i}^{(s)} - \lambda_{,i}^{s}M_0)X_0$$
$$X_{,i}^{(t)} = -S_o^{-1}(K_{,i}^{(t)} - \lambda_0 M_{,i}^{(t)} - \lambda_{,i}^{(t)}M_0)X_0 \quad (21)$$

$$Var_{(L)}(X) =$$
$$-S_o^{-1}\left(K_{,i}^{(s)} - \lambda_{,i}^{(s)}M_0\right)X_0X_0^T\left(K_{,j}^{(s)} - \lambda_{,j}^{(s)}M_0\right)^T S_o^{-T}R_{ss}(\xi_{ij})$$
$$+S_o^{-1}\left(\lambda_o M_i^{(t)} + \lambda_i^{(t)}M_0\right)X_0X_0^T\left(\lambda_o M_{,j}^{(t)} + \lambda_j^{(t)}M_0\right)^T S_o^{-T}R_{ss}(\xi_{ij})$$
$$-S_o^{-1}(K_{,i}^{(s)} - \lambda_i^{(s)}M_0)X_0X_0^T(\lambda_o M_{,j}^{(t)} + \lambda_{,j}^{(t)}M_0)S_o^{-T}R_{ss}(\xi_{ij}) -$$
$$S_o^{-1}(\lambda_o M_{,j}^{(t)} + \lambda_{,j}^{(t)}M_0)X_0X_0^T(K_{,i}^{(s)} - \lambda_i^{(s)}M_0)^T S_o^{-T}R_{ss}(\xi_{ij}) \quad (22)$$

where $R_{ss}(\xi_{ij})$ is the autocorrelation, $R_{st}(\xi_{ij})$ is the cross-correlation, $\gamma_{E\rho}$ is the coefficient correlation, and d is the correlation distance.

$$R_{ss}(\xi_{ij}) = \sigma_E^2\, exp\left(-\frac{\xi^2}{4d^2}\right) \quad (23)$$

The coefficient of variation (COV) of the response eigenvector shows the degree of dispersion of the distribution of the eigenvectors and is given as follows:

$$\text{COV} = \frac{\sigma_X}{|\mu_X|} \quad (24)$$

where σ_X is the standard deviation of the random eigenvectors and μ_X is the mean of the random eigenvectors.

IV. NUMERICAL EXAMPLES

The constituent geometric dimensions of the FGM beams were h = 1 m and L = 20 m. Young's modulus ratio is E_0/E_1, where E_0 and E_1 denote Young's modulus at the middle and the top surface of the FGM beam, respectively. The mid-surface of the beam is 100% aluminum, with mean values of material parameters similar to [31]: E_0 = 70 GPa, ρ_0 = 2,780 kg/m^3, and ν=0.33. Using the first-order perturbation expansions of the proposed stochastic FEM analysis, the response variability of the eigenvector of the first mode was calculated and is shown in Figures 2, 3, and 4. The results of the MCS using 10,000 samples generated by representative spectral [32] are given also. As seen in Figures 2-4, the perturbation method results agree very well with the MCS [30] results in all cases of mean, Standard Deviation (SD), and COV of eigenvector.

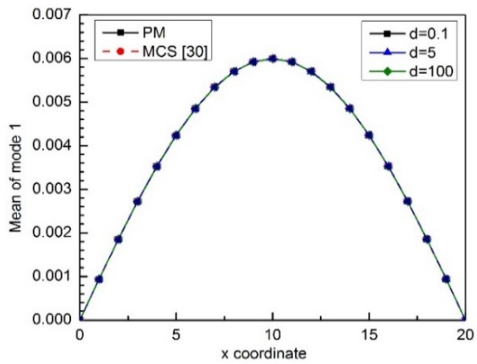

Fig. 2. Mean of eigenvector.

Figure 5 shows the effect of the correlation distance d of the stochastic field on the variability of the COV of the eigenvector, where the results of the proposed formulation are compared with those of the MCS for the same cases of the stochastic FEM. The results designated by a dotted line denote the corresponding results of the MCS for correlation distances of 1, 10, 15, and 20. As seen in Figure 5, the results of the perturbation method agree very well with the results of the MCS in all cases. It is observed that the increasing rate of the correlation distances is accelerated with a decrease in the COV of the stochastic field.

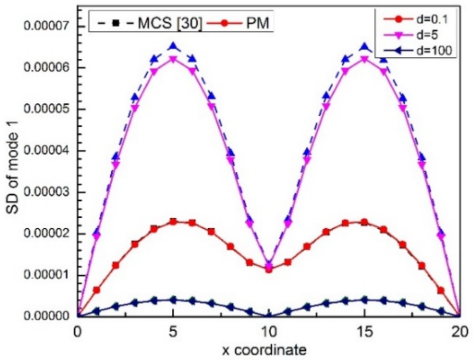

Fig. 3. SD of eigenvector.

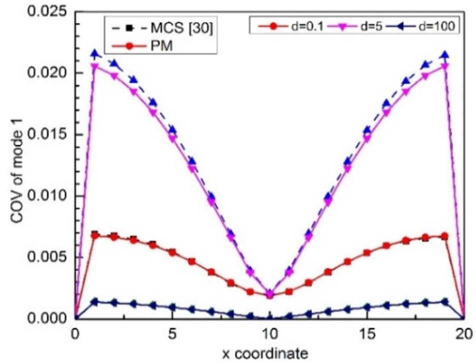

Fig. 4. COV of eigenvector.

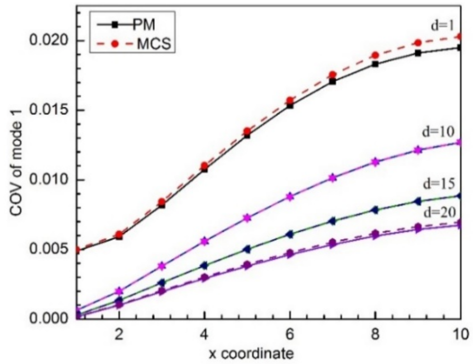

Fig. 5. COV of eigenvector as *x* coordinate of beam for different correlation distance *d*.

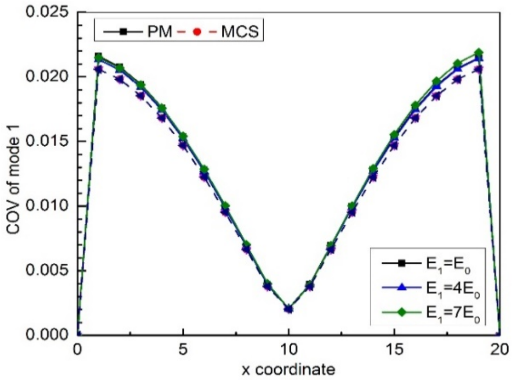

Fig. 6. Effect of elastic modulus ratio on the COV of the eigenvector.

Figure 6 shows the effect of Young's modulus ratio on the COV of the eigenvector. The COV of the eigenvectors is observed to be unaffected by the modulus ratio of the Young parameter. The elastic modulus ratio affects only the specified stiffness of the beam and does not affect the random field properties, which are assumed to be unidirectionally varying along the axis beams.

V. CONCLUSIONS

This paper extended the stochastic FEM using the first-order perturbation approach to develop the eigenvector of a beam. In the stochastic FEM approach, the local average scheme was used to discretize random processes rather than the midpoint rule. The comparison of the mean, standard deviation, and coefficient of variation of the eigenvector in an FGM beam predicted by stochastic FEM with those by MCS in previous studies considering random Young's modulus produced the following conclusions for the systems:

- The FGM model was analyzed to demonstrate the adequacy of the proposed formulation. The first-order perturbation results showed good agreement with the MCS results for mean, standard deviation, and coefficient of variation of the eigenvector.
- The increasing rate of the correlation distances was accelerated with a decrease in the coefficient of variation in both FEM and Monte Carlo simulation.
- For both analysis methods, the results showed that the response variability of the eigenvector was not affected by either the elastic modulus ratio on the top or middle surfaces of the FGM beams.

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