NURBS-based Isogeometric Analysis and Refined Plate Theory Application on a Functionally Graded Plate Subjected to Random Loads

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ABSTRACT

In structural design standards, loads are often assumed to be random quantities to give load factors. This study deals with the Stochastic Isogeometric Analysis (SIGA) for a Functionally Graded Plate (FGP) subjected to random distribution loads. The spatial random variation of distribution loads is modeled as a homogeneous Gaussian random field in the plane of the functionally graded plate. The governing equation of the functional grade plate is derived using the NURBS-based isogeometric analysis and the refined plate theory. SIGA is developed based on standard NURBS-based isogeometric analysis in conjunction with the first-order perturbation expansions of random loads. This approach was verified with Monte Carlo simulation, and the numerical results showed the effect of random loads on the variation of displacements and stresses of the functionally graded plate.

Keywords-SIGA; random loads; functionally graded plate; random field

I. INTRODUCTION

Functionally Graded Material (FGM) is a special composite material. Typical FGMs are made of metal and ceramic material components distributed to vary continuously with position along the thickness direction. FGMs are used in mechanical engineering and heavy industry [1-3]. In recent years, studies have investigated structures made from FGMs, such as functionally graded beams [4-8] and functionally graded plates [9-11]. In deterministic structure problems, such as the analysis of beams [12-15], frame structures [16, 17], and plates [18-21], the input data are deterministic, so it is easier to solve than stochastic problems. Several studies investigated the stochastic structure's problem [22-24] and the stochastic finite element method [25]. In [26], the fluctuation of the eigenvalue of free vibration of non-uniform beams was studied using the stochastic finite element method, considering the randomness of the elastic modulus. In [27], the perturbation technique was applied to develop a stochastic finite element method for the buckling of a functionally graded plate with uncertain material properties in thermal environments. In [28-30], the static and dynamic responses of a beam resting on a foundation or elastic support were analyzed, considering various random parameters. In [31], a formulation was proposed to determine the response variability in a plate structure due to the randomness of the Poisson ratio using the weighted integral method.

Recently, in addition to the stochastic finite element method, several studies used stochastic isogeometric analysis at uncertain structures. In [32], the Karhunen–Loève expansion was used to develop the spectral stochastic isogeometric analysis for problems of linear elasticity. In [33], the Galerkin isogeometric method was used to solve the Fredholm integral eigenvalue problem. In [34, 35], stochastic isogeometric analysis was proposed for functionally graded plates considering the random field of material properties. This study focuses on developing the stochastic isogeometric analysis for the static bending problem of a functionally graded plate under random loads.

II. FORMULATION OF STOCHASTIC ISOGEOMETRIC ANALYSIS

A. The Non-Uniform Rational B-Spline (NURBS) Functions

The Non-Uniform Rational B-Spline (NURBS) functions are defined via the B-spline basic functions [11, 36-37]. To construct a set of n B-spline basic functions of order p, a knot vector Ξ is defined as a set of coordinates in the parametric space in one-dimensional parametric domain $\xi \in [0,1]$:

$$\Xi = \{\xi_1, \xi_2, \ldots, \xi_{n+p+1}\} \quad \xi_i \le \xi_{i+1},\ i = 1,2,\ldots,n+p \tag{1}$$

where $\xi_i \in R$ is the i-th knot, i is the knot index, and p is the polynomial order. Given the knot vector, the B-spline basis functions are defined recursively starting with piecewise constants (p=0, 1, 2, 3..) as follows:

For $p = 0$:

$$N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \leqslant \xi < \xi_{i+1}, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

For $p = 1,2,3,\ldots$:

$$N_{i,p}(\xi) = \frac{\xi-\xi_i}{\xi_{i+p}-\xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1}-\xi}{\xi_{i+p+1}-\xi_{i+1}} N_{i+1,p-1}(\xi) \tag{3}$$

Taking a linear combination of B-spline basis functions constructs B-spline curves where the coefficients of the basis functions are referred to as control points:

$$C(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi) B_i \tag{4}$$

where B_i are the control points. The B-spline surfaces are defined by the tensor product of basis functions in two parametric dimensions ξ and η with two-knot vectors:

$$S(\xi,\eta) = \sum_{i=1}^{n}\sum_{j=1}^{m} N_{i,p}(\xi) M_{j,q}(\eta) P_{i,j} = \sum_{I=1}^{m\times n} N_I^b(\xi,\eta) P_I \tag{5}$$

where $N_I^b(\xi,\eta) = N_{i,p}(\xi) M_{j,q}(\eta)$ is the shape function associated with control point I.

Non-Uniform Rational B-Splines are defined based on the B-splines by adding an individual weight w_I:

$$R_{i,p}(\xi) = \frac{N_{i,p}(\xi) w_i}{\sum_{I=1}^{n} N_{I,p}^b(\xi) w_I} \tag{6}$$

The NURBS surface is constructed by combining the rational basis functions and the coefficient at control points B_i:

$$S(\xi,\eta) = \sum_{I=1}^{m\times n} R_I(\xi,\eta) B_I \tag{7}$$

with $R_I(\xi,\eta) = \dfrac{N_I^b w_I}{\sum_{I=1}^{m\times n} N_I^b(\xi,\eta) w_I}$

B. NURBS-Based Isogeometric Formulations for a Functionally Graded Plate Based on Refined Plate Theory

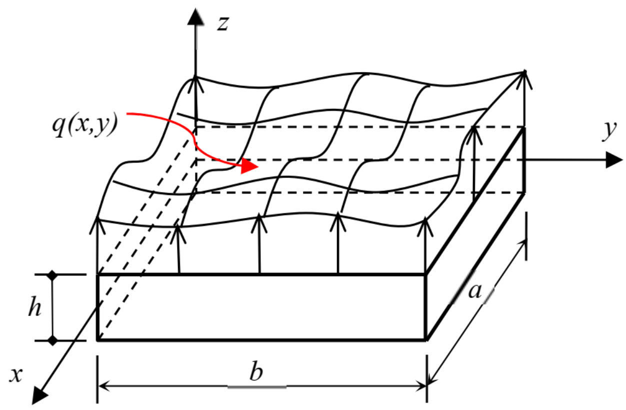

Fig. 1. Random field of distribution loads on plate.

Consider a functionally graded plate subjected to random loads, with the coordinate system placed at the mid-plane of the plate as shown in Figure 1. The plate displacement fields were calculated using the refined plate theory [38], which allows taking into account the shear deformation effect, and are used to formulate the governing equations as follows:

$$\begin{aligned} U &= u_0 - z\frac{\partial w_b}{\partial x} + \left[\frac{1}{4}z - \frac{5}{3}z\left(\frac{z}{h}\right)^2\right]\frac{\partial w_s}{\partial x} \\ V &= v_0 - z\frac{\partial w_b}{\partial y} + \left[\frac{1}{4}z - \frac{5}{3}z\left(\frac{z}{h}\right)^2\right]\frac{\partial w_s}{\partial y} \\ W &= w_b + w_s \end{aligned} \tag{8}$$

The linear strains can be obtained by differentiating (8) as:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix} + z\begin{Bmatrix} -\dfrac{\partial^2 w_b}{\partial x^2} \\ -\dfrac{\partial^2 w_b}{\partial y^2} \\ -2\dfrac{\partial^2 w_b}{\partial x \partial y} \end{Bmatrix} + \Psi\begin{Bmatrix} -\dfrac{\partial^2 w_s}{\partial x^2} \\ -\dfrac{\partial^2 w_s}{\partial y^2} \\ -2\dfrac{\partial^2 w_s}{\partial x \partial y} \end{Bmatrix} \tag{9}$$

$$\begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \chi \begin{Bmatrix} \dfrac{\partial w_s}{\partial y} \\ \dfrac{\partial w_s}{\partial x} \end{Bmatrix} \tag{10}$$

where:

$$\Psi(z) = -\frac{1}{4}z + \frac{5}{3}z\left(\frac{z}{h}\right)^2 \ , \ \chi(z) = 5\left[\frac{1}{4} - \left(\frac{z}{h}\right)^2\right] \quad (11)$$

Young's modulus E of the functionally graded plate is assumed as:

$$E(z) = (E_c - E_m)\left(\frac{z}{h} + \frac{1}{2}\right)^p + E_m \quad (12)$$

The linear constitutive relation of a plate is given by formulation:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} \quad (13)$$

where:

$$\begin{aligned} Q_{11} &= \frac{E(z)}{1-v^2} \\ Q_{22} &= Q_{11} \\ Q_{12} &= \frac{vE(z)}{1-v^2} \\ Q_{44} &= G_{23} \\ Q_{55} &= G_{13} \\ Q_{66} &= G_{12} \end{aligned} \quad (14)$$

The displacement field $\widehat{U}$ of the plate is approximated via the NURBS basis functions:

$$\widetilde{U}(\xi,\eta) = \sum_{i=1}^{m\times n} R_i(\xi,\eta)U_i \quad (15)$$

where U_i is the vector of nodal degrees of freedom associated with the control point *i*:

$$U_i = \begin{Bmatrix} u_0 \\ v_0 \\ w_0^b \\ w_0^s \end{Bmatrix}_i \quad (16)$$

The governing equation of the functionally graded plate is derived using the virtual work to be:

$$\int_\Omega \varepsilon_b^T D^b \,\delta\varepsilon_b d\Omega + \int_\Omega \gamma^T A^s \,\delta\gamma d\Omega = \int_\Omega q\delta w d\Omega \quad (17)$$

where the stiffness matrices A^s and D^b are given as follows:

$$D^b = \begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & B_{12} & 0 & B_{11}^s & B_{12}^s & 0 \\ A_{12} & A_{22} & 0 & B_{12} & B_{22} & 0 & B_{12}^s & B_{22}^s & 0 \\ 0 & 0 & A_{66} & 0 & 0 & B_{66} & 0 & 0 & B_{66}^s \\ B_{11} & B_{12} & 0 & D_{11} & D_{12} & 0 & D_{11}^s & D_{12}^s & 0 \\ B_{12} & B_{22} & 0 & D_{12} & D_{22} & 0 & D_{12}^s & D_{22}^s & 0 \\ 0 & 0 & B_{66} & 0 & 0 & D_{66} & 0 & 0 & D_{66}^s \\ B_{11}^s & B_{12}^s & 0 & D_{11}^s & D_{12}^s & 0 & H_{11}^s & H_{12}^s & 0 \\ B_{12}^s & B_{22}^s & 0 & D_{12}^s & D_{22}^s & 0 & H_{12}^s & H_{22}^s & 0 \\ 0 & 0 & B_{66}^s & 0 & 0 & D_{66}^s & 0 & 0 & H_{66}^s \end{bmatrix} \quad (18)$$

$$A^s = \begin{bmatrix} A_{44}^s & 0 \\ 0 & A_{55}^s \end{bmatrix} \quad (19)$$

where A_{ij}, B_{ij} etc are parameters of the plate stiffness, defined by:

$$\begin{aligned} \{A_{ij}, B_{ij}, D_{ij}\} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \{1, z, z^2\}\, Q_{ij} dz,\ (i = 1,2,6) \\ B_{ij}^s &= \int_{-h/2}^{h/2} \psi\, Q_{ij} dz,\ (i = 1,2,6) \\ D_{ij}^s &= \int_{-h/2}^{h/2} z\,\psi Q_{ij} dz,\ (i = 1,2,6) \\ H_{ij}^s &= \int_{-h/2}^{h/2} \psi^2\, Q_{ij} dz,\ (i = 1,2,6) \\ A_{ij}^s &= \int_{-h/2}^{h/2} \chi\, Q_{ij} dz,\ (i = 4,5) \end{aligned} \quad (20)$$

The governing equation for bending the functionally graded plate is in the following form:

$$KU = F \quad (21)$$

where *K*, *U,* and *F* denote the stiffness matrix, generalized displacement, and force vector. The global stiffness matrix *K* is given by:

$$K = \int_\Omega \left\{ \{B^0 \quad B^{bb} \quad B^{bs}\} D_b \begin{Bmatrix} B^0 \\ B^{bb} \\ B^{bs} \end{Bmatrix} + (B^s)^T A^s B^s \right\} d\Omega \quad (22)$$

and the load vector is given by:

$$F = \int_\Omega q(x,y) \begin{Bmatrix} 0 \\ 0 \\ R_i \\ R_i \end{Bmatrix} d\Omega \quad (23)$$

III. STOCHASTIC ISOGEOMETRIC ANALYSIS

The random field of distribution loads is assumed as a homogeneous Gaussian random field in the plane of the plate:

$$q(x,y) = q_0[1 + f(x,y)] \quad (24)$$

where *f(x,y)* is a homogeneous random field with zero mean. The auto-correlation function of the random field *f(x,y)* is assumed as:

$$R(\xi_x, \xi_y) = \sigma^2\, exp\left(-\frac{|\xi_x| + |\xi_y|}{d}\right) \quad (25)$$

where ξ_x, ξ_y are components of the separation vector ξ between two points in the domain of the plate, and σ and d are the coefficient of variation and the correlation distance of the random field, respectively. In this study, the value of the random field in elastic modulus was discretized as a point method by approximation at Gauss points. The set of random variables follows the statistical properties of the random fields:

$$f = \{f_1, f_2, \dots f_N\} \quad (26)$$

Taking the Taylor series expansion at *f*=0 based on the assumption that all components of the random field *f* are small, gives:

$$F(f) = F_0 + \sum_{i=1}^{N} \frac{\partial F}{\partial f_i} f_i + \frac{1}{2}\sum_{i,j=1}^{N} \frac{\partial^2 F}{\partial f_i \partial f_j} f_i f_j + \cdots$$

$$U(f) = U_0 + \sum_{i=1}^{N} \frac{\partial U}{\partial f_i} f_i + \frac{1}{2}\sum_{i,j=1}^{N} \frac{\partial^2 U}{\partial f_i \partial f_j} f_i f_j + \ldots \quad (27)$$

For the zeroth perturbation equation:

$$K_0 U_0 = F_0 \quad (28)$$

For the first-order perturbation equation:

$$K_0 \frac{\partial U}{\partial f_i} = \frac{\partial F}{\partial f_i} \quad (29)$$

The first-order perturbation solutions are:

$$\mu_U = U_0$$

$$Cov_U = \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial U}{\partial f_i} \frac{\partial U}{\partial f_j} R_{ij} \quad (30)$$

where:

$$\mu_U = E(U)$$

$$\mathrm{Cov}_U = E\left[(U - \mu_U)(U - \mu_U)^T\right] \quad (31)$$

denote the mean and variance of the displacement U, respectively, R_{ij} is the covariance matrix of the random variable f, and N is the number of random variables. As U represents nodal parameters at control points, it is not the real displacements at the nodes. So, The mean vector and covariance matrix, using the relation between U and $\widetilde{U}$ in (15) can be obtained as:

$$\mu_{\widetilde{U}} = \Phi \mu_U$$

$$Cov_{\widetilde{U}} = \Phi Cov_U \Phi^T \quad (32)$$

where Φ is the transformation matrix between U and $\widetilde{U}$. The response variability is a ratio of the standard deviation of displacement to the absolute mean eigenvalue as:

$$COV = \frac{\sqrt{Var_{\widetilde{U}}}}{|\mu_{\widetilde{U}}|} \quad (33)$$

IV. NUMERICAL EXAMPLES

A. *Example 1: Numerical Tests*

To evaluate the proposed stochastic isogeometric analysis, this method is compared with the crude Monte Carlo simulation in [39] using the spectral representation method [40]. The random sample function of the Gaussian random fields in (24) can be represented by the cosine series as follows:

$$f(x,y) = \sqrt{2}\sum_{n_1=0}^{N_1}\sum_{n_2=0}^{N_2}\begin{bmatrix} A_{n_1 n_2}^{(1)} cos\left(\omega_{1_{n1}} x + \omega_{2_{n2}} y + \varphi_{n_1 n_2}^{(1)}\right) \\ A_{n_1 n_2}^{(2)} cos\left(\omega_{1_{n1}} x - \omega_{2_{n2}} y + \varphi_{n_1 n_2}^{(2)}\right)\end{bmatrix} \quad (34)$$

The phase angles $\varphi_{n_1 n_2}^{(1)}, \varphi_{n_1 n_2}^{(2)}$ are uniformly distributed in the range of [0, 2π].

The simply supported rectangular plate was considered subjected to random loads as shown in Figure 2. The elastic modulus and the Poisson ratio were set as E_0=200×10^{5}MPa and v=0.30, respectively. The thickness of the plate was t=10mm, and the coefficient of variation of the random field σ was 0.1. The mean of random loads is unit distributed load.

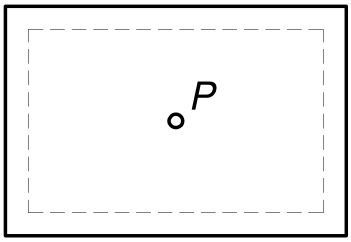

Fig. 2. Simply supported rectangular plate.

Figure 3 shows the response coefficient of the variation of displacement at the center point P of the plate, obtained by the present approach (SIGA) and Monte Carlo simulation [39] with 10000 samples.

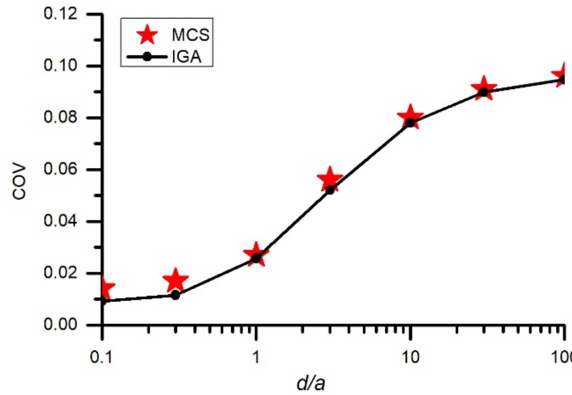

Fig. 3. Response COV of displacement at P as a function of correlation distanced d.

Figure 3 shows a good agreement between the SIGA and the Monte Carlo simulation. The coefficient of variation of the displacement trends to the coefficient of variation of the random field of random loads.

B. *Example 2*

A simply supported square functionally graded plate, made of aluminum and alumina (Al/Al2O3), was considered. Material properties were similar to [41], E_m=70MPa and E_c=380MPa and the Poisson ratio was set to 0.3. The plate was subjected to random loads. The mean of random loads was unit distributed load and the power index p was set to 1. Figure 4 shows the COV response with various values of COV for the random fields of random loads obtained for the plane-normal displacement at the center point P of the plate. The response COV is bigger if the COV of the random field is larger. Also, the response COV increases when the coefficient of variation of the random field increases and approaches the coefficient of variation of the random field σ in all cases.

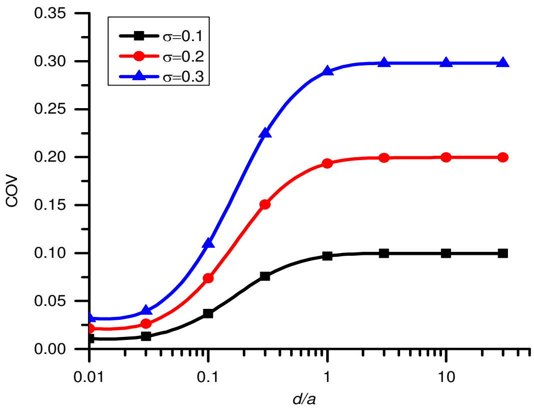

Fig. 4. Response COV as a function of correlation distanced.

V. CONCLUSION

This study successfully developed the Stochastic Isogeometric Analysis (SIGA) for a functionally graded plate under random loads with the assumption that random loads are a homogeneous Gaussian random field in the plate plane. The coefficient of variation of the deflection in the center of the plate predicted by SIGA was validated with the results of Monte Carlo simulation. The numerical examples showed good agreement between the COV of displacement predicted by the present study and the Monte Carlo simulation. Also, the numerical examples clearly show that the randomness of load affects significantly the response of the plate. The effect of the correlation distance on the COV of displacement is clear, and the response COV increases when the correlation distances increase.

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