Finite Element Analysis of a Continuous Sandwich Beam resting on Elastic Support and Subjected to Two Degree of Freedom Sprung Vehicles

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ABSTRACT

This paper has developed a Finite Element Method (FEM) to calculate the dynamic response of a continuous sandwich beam resting on elastic support subjected to moving vehicles. The equation of motion is derived using the classical beam theory and FEM. The vehicle model is a two Degree of Freedom (2DOF) system that moves with a constant velocity. The governing equation of motion is integrated by applying the Wilson-θ time integration method to obtain the dynamic response in each time step. Numerical examples investigate the displacement of the sandwich beam with various values of the structure and vehicle velocity. The effects of the stiffness of elastic support and the vehicle velocity on displacement are studied.

Keywords-FEM; forced vibration; continuous beam; elastic support

I. INTRODUCTION

Structures in the construction industry can be multiform and diverse. The most commonly used structures in civil engineering are columns [1, 2], piles [3], beams [4-8], frames [9, 10], and plates [11-19]. In general, a beam is a flexural member and the normal stress has maximum values at the top and the bottom of the beam. So, sandwich beams will have optimal bearing capacity if the outer layer is a high-strength material. In practice, the fibers at the top and bottom of the beam can be made of steel and the core layer from concrete or wood. If the sandwich beam is made of a suitable material, it will have a better load capacity, reduced height, and a larger span. Thus, the sandwich beam has many advantages if it is made of appropriate materials. A sandwich beam usually has two layers, i.e. the outer layer and the core layer, as shown in Figure 1.

Structures are subjected to various dynamic loads, e.g. from wind, moving vehicles, impulsive loading, etc. Structure dynamics have many practical aspects, e.g. the dynamic behavior of beams [20-23] and the dynamics of plates [24, 25]. The dynamic response of an elastic plate in a viscoelastic medium, resting on a viscoelastic Winkler foundation is investigated in [26]. The forced vibration of functionally graded plates resting on a viscoelastic elastic foundation was solved analytically in [27]. The effect of randomness in the elastic modulus on the eigenvalue of free vibration of non-uniform beams is studied using stochastic finite elements in [28]. The variability dynamic response of a beam subjected to a moving load with various uncertain parameters is investigated by Monte Carlo simulation in [29]. The dynamic response of a beam resting on a viscoelastic foundation was analyzed with a novel state-space formulation considering the interaction of the oscillator-beam-foundation system in [30]. The dynamic response of sandwich beams with a viscoelastic core subjected to moving loads was studied by FEM in [31]. An analytical solution was preformed to investigate the transient response of sandwich beams in [32]. A new semi-analytical method was applied in [33] to study the dynamics of beams. The transient response of double beams with viscoelastic boundary conditions under a moving load was investigated by the analytical method in [34]. The nonlinear dynamic response of the coupled vehicle-pavement system was computed using the Galerkin truncation method in [35], in which the pavement was modeled as a Timoshenko beam on the six-parameter foundation. The dynamic response of bridges to moving vehicle loads was investigated using a semi-analytical approach in [36]. The arch bridge subjected to the moving loads was modeled as a continuous beam resting on interconnected springs to compute the dynamic response using the analytical method [37]. Author in [38] presented an analytical and numerical method to study the dynamics of a beam–mass system. Experiments and finite element simulations studied the dynamic response of metallic sandwich beams under impact loading in [39].

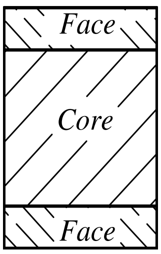

Fig. 1. Cross section of a sandwich beam.

In this study, FEM is used to study the forced vibration of a continuous sandwich beam resting on an elastic support.

II. FINITE ELEMENT FORMULATION

Let us consider a continuous sandwich beam resting on elastic supports, subjected to moving vehicles, with the coordinate system placed at the mid-plane of the beam as shown in Figure 2.

The deflection w of the beam can be written in a matrix form as [40]:

$$w\left(x,t\right)=\left\{N_1\left(x\right)\quad N_2\left(x\right)\quad N_3\left(x\right)\quad N_4\left(x\right)\right\}\begin{Bmatrix} w_1\left(t\right)\\ \theta_1\left(t\right)\\ w_2\left(t\right)\\ \theta_2\left(t\right)\end{Bmatrix}=\left\{N\left(x\right)\right\}\left\{q\left(t\right)\right\}_e \quad (1)$$

where $\left\{q\left(t\right)\right\}_e$ denotes the displacement vector of the finite element and $\left\{N\left(x\right)\right\}$ are Hermite interpolation functions, defined as follows:

$$\begin{cases} N_1\left(x\right)=1-3\dfrac{x^2}{L_e^2}+2\dfrac{x^3}{L_e^3};\ N_2\left(x\right)=x\left(1-2\dfrac{x}{L_e}+\dfrac{x^2}{L_e^2}\right)\\ N_3\left(x\right)=3\dfrac{x^2}{L_e^2}-2\dfrac{x^3}{L_e^3};\ N_4\left(x\right)=x\left(-\dfrac{x}{L_e}+\dfrac{x^2}{L_e^2}\right)\end{cases} \quad (2)$$

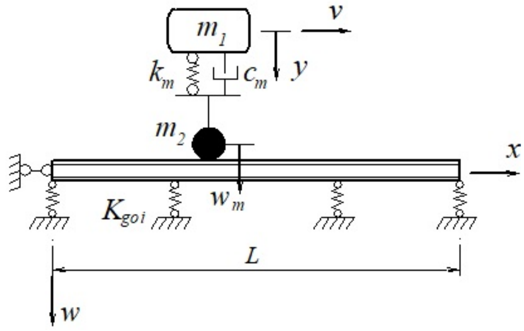

Fig. 2. Continuous sandwich beam resting on elastic supports and subjected to a moving vehicle.

The strain energy of the beam element is:

$$U_e=\frac{1}{2}\int_0^{L_e}\left\{\underbrace{\int_A\left\{E_{Face}\left[y\frac{\partial^2 w_0}{\partial x^2}\right]^2\right\}dA}_{Face}+\underbrace{\int_A\left\{E_{Core}\left[y\frac{\partial^2 w_0}{\partial x^2}\right]^2\right\}dA}_{Core}\right\}dx \quad (3)$$

The potential energy of the elastic support is defined as:

$$U_s=\frac{1}{2}\sum k_s\left[w_0\left(x\right)\right]^2 \quad (4)$$

where k_s is the stiffness of the elastic support.

The kinetic energy of the beam element is given as:

$$\Pi_e=\frac{1}{2}\int_0^{L_e}\left\{\underbrace{\int_A\left\{\rho_{Face}\left[\frac{\partial w_0}{\partial t}\right]^2\right\}}_{Face}+\underbrace{\int_A\left\{\rho_{Core}\left[\frac{\partial w_0}{\partial t}\right]^2\right\}}_{Core}\right\}dx \quad (5)$$

The governing equation of the sandwich beam is derived as:

$$\left[M^b\right]\{\ddot{w}\}+\left(\left[K^b\right]+\left[K^s\right]\right)\{w\}=\{N\}^T f_w \qquad (6)$$

where $\left[K^s\right],\left[K^b\right],\left[M^b\right]$ are the stiffness matrix of the elastic support and the stiffness and mass matrices of the sandwich beam, respectively.

The equation of motion of the moving mass m_1 and the contact force f_w are determined as follows:

$$m_1\ddot{y}+c_m\left(\dot{y}-\dot{w}_m\right)+k_m\left(y-w_m\right)=0 \qquad (7)$$

$$f_w=\left(m_1+m_2\right)g-m_2\ddot{w}_m-m_1\ddot{y} \qquad (8)$$

where w_m is the vertical dynamic deflection of the beam at mass m_2.

Combining (6) to (8), the governing equation is obtained:

$$\begin{bmatrix}[M]^b+m^* & [N]^T m_1\\ 0 & m_1\end{bmatrix}\begin{bmatrix}\{\ddot{w}\}\\ \ddot{y}\end{bmatrix}+\begin{bmatrix}c^* & 0\\ -c_m[N] & c_m\end{bmatrix}\begin{bmatrix}\{\dot{w}\}\\ \dot{y}\end{bmatrix}+\begin{bmatrix}[K]^b+[K]^s+k^* & 0\\ -c_m\dot{x}\left[\dot{N}\right]_x-k_m[N] & k_m\end{bmatrix}\begin{bmatrix}\{w\}\\ y\end{bmatrix}=\begin{bmatrix}[N]^T(m_1+m_2)g\\ 0\end{bmatrix} \qquad (9)$$

where:

$$\begin{aligned}m^*&=m_2[N]^T[N]\\ c^*&=2m_2\dot{x}(t)[N]^T[N]_x\\ k^*&=m_2\dot{x}^2(t)[N]^T[N]_{xx}+m_2\ddot{x}(t)[N]^T[N]_x\end{aligned} \qquad (10)$$

III. SOLVING THE EQUATION OF MOTION WITH THE WILSON-θ METHOD

The equation of motion of the vehicle-bridge interaction can be written in the following general form:

$$[M]\{\ddot{u}(t)\}+[C]\{\dot{u}(t)\}+[K]\{u(t)\}=\{F(t)\} \qquad (11)$$

where $[M],[C],[K],\{F(t)\}$ are the mass, damping, and stiffness matrices, and the force vector, respectively. In order to use the Wilson scheme, we introduce the effective stiffness matrix as form:

$$\left[\tilde{K}\right]=[K]+a_0[M]+a_1[C] \qquad (12)$$

where: $a_0=\dfrac{1}{\beta(\theta\Delta t)^2};\ a_1=\dfrac{\gamma}{\beta\theta\Delta t}$ (13)

The vector of effective forces at time $t+\theta$ can be calculated as follows:

$$\begin{aligned}\left\{\tilde{F}(t+\theta)\right\}&=\{F(t)\}+\theta\left(\{F(t+\theta)\}-\{P(t)\}\right)+\\ &+[C]\left(a_1\{u(t)\}+a_4\{\dot{u}(t)\}+a_5\{\ddot{u}(t)\}\right)\\ &+[M]\left(a_0\{u(t)\}+a_1\{\dot{u}(t)\}+a_2\{\ddot{u}(t)\}\right)\end{aligned} \qquad (14)$$

where $a_2=\dfrac{1}{\beta\theta\Delta t};\ a_4=\dfrac{\gamma}{\beta}-1;\ a_5=\left(\dfrac{\gamma}{\beta}-2\right)\dfrac{\theta\Delta t}{2}$. The values of the parameters are selected as $\gamma=1/2,\ \beta=1/6$, and $\theta\geq 1.37$.

The following linear system of equations computes the displacements at time after initialization of displacement, velocity, and acceleration vectors:

$$\left[\hat{K}\right]\{u(t+\theta)\}=\left\{\tilde{F}(t+\theta)\right\} \qquad (15)$$

The vector of acceleration at the time $t+\theta$ is calculated by:

$$\begin{aligned}\{\ddot{u}(t+\theta)\}&=a_0\left(\{u(t+\theta)\}-\{u(t)\}\right)\\ &-a_2\{\dot{u}(t)\}-a_3\{\ddot{u}(t)\}\end{aligned} \qquad (16)$$

where $a_3=\dfrac{1}{2\beta}-1$.

Acceleration, velocity, and displacement at time $t+\Delta t$ are calculated from:

$$\begin{cases}\{\ddot{u}(t+\Delta t)\}=\{\ddot{u}(t)\}+\dfrac{1}{\theta}\left(\{\ddot{u}(t+\theta)\}-\{\ddot{u}(t)\}\right)\\ \{\dot{u}(t+\Delta t)\}=\{\dot{u}(t)\}+a_6\{\ddot{u}(t)\}+a_7\{\ddot{u}(t+\Delta t)\}\\ \{u(t+\Delta t)\}=\{u(t)\}+\Delta t\{\dot{u}(t)\}+a_8\{\ddot{u}(t)\}\\ \qquad +a_9\{\ddot{u}(t+\Delta t)\}\end{cases} \qquad (17)$$

where: $a_6=(1-\gamma)\Delta t;\ a_7=\gamma\Delta t;\ a_8=\left(\dfrac{1}{2}-\beta\right)\Delta t^2;\ a_9=\beta\Delta t^2$.

IV. NUMERICAL EXAMPLES

A. *Example 1*

Considering the continuous sandwich beam shown in Figure 3, subjected to a moving vehicle with 2DOF. The rectangular sandwich beam has 3 uniform spans of 4m, the dimensions of cross-section are h=60cm, b=25cm, and the thickness of the face layer is 1cm. The parameters for the vehicle in Figure 3 are: m_1=1680kg and m_2=840kg. The structural properties of the sandwich beam are given in Table I.

TABLE I. MATERIAL PROPERTIES OF THE SANDWICH BEAM

Fiber	Material properties		
	Material	**Young's modulus (GPa)**	**Mass density (kg/m^3)**
Face layer	Steel	200	7800
Core	Wood	10	700

Figure 4 shows the displacement at the center of the beam for 2 cases of stiffness of elastic support. The speed of the vehicle is set to 5, 10, and 15m/s. The Figure clearly shows that the displacements at the center of the beam are highly similar. So, the effect of speed of the vehicle on the displacement of the beam is small.

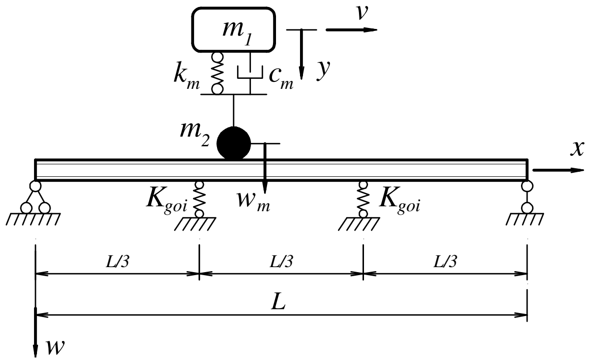

Fig. 3. Three-span sandwich beam subjected to a moving vehicle.

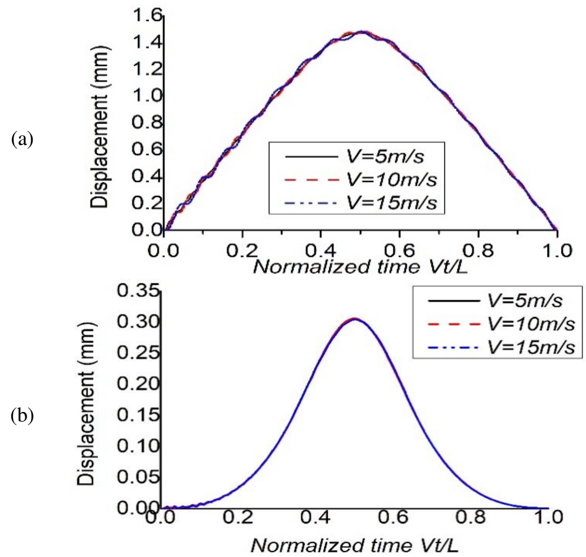

Fig. 4. Displacement at the center of the beam for K_{goi} equal to (a) 10^7, (b) 10^8 N/m.

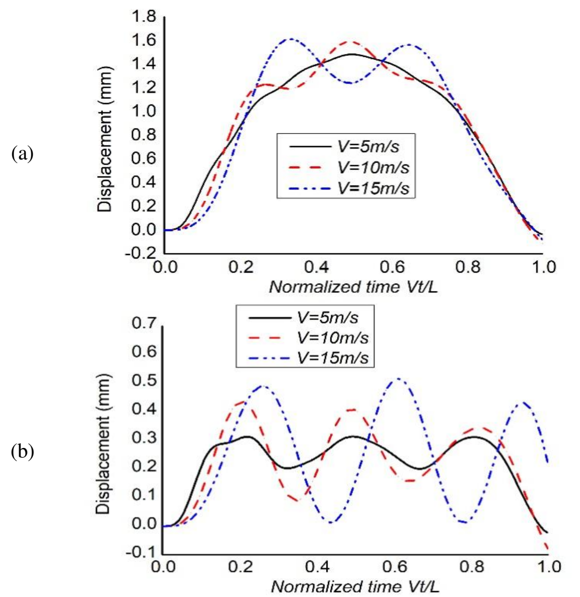

Fig. 5. Displacement at the contact point with the m_2 mass of the vehicle for K_{goi} equal to (a) 10^7, (b) 10^8 N/m.

The displacement of the mass of the vehicle m_2 is investigated with vehicle speed values of 5, 10, and 15m/s and stiffness of the elastic support K_{goi} equal to 10^7 and 10^8N/m. When the speed is slow, the displacement is close to the static displacement. At high vehicle speed, the displacement is fluctuated around the displacement line at low speed.

Figure 6 illustrates the displacement at the center of the beam and at the contact point m_2 of the moving vehicle with speed of 10m/s for 3 stiffening cases of elastic support. As the stiffness of the support increases, the stiffness of the structure increases, leading to a decrease in displacement. When the stiffness of the support is quite large, the displacement at the center of the beam has an almost sinusoidal shape. It is clear that the stiffness of elastic support affects the displacement of the beam.

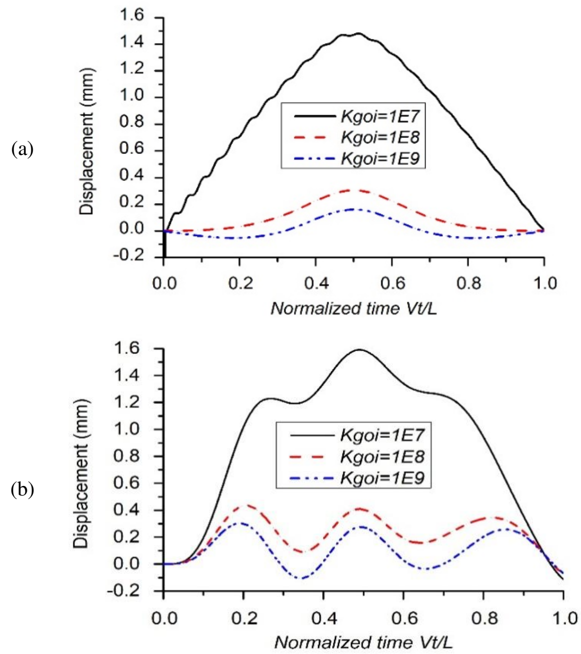

Fig. 6. Displacements for 3 stiffening cases of elastic support: (a) at the center of the beam, (b) at the contact point.

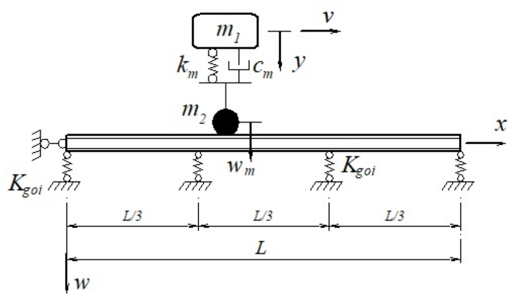

Fig. 7. Displacements of sandwich beam resting on elastic support.

B. Example 2

In this example, the dynamics of a sandwich beam is investigated by replacing two rigid supports of the example 1 with elastic supports as described in Figure 7. The displacements at the center of the beam and at the contact point are shown in Figure 8 for 3 stiffening cases of elastic support. Figure 8 clearly shows that the displacements are greatly increased. It also shows the strong influence of the stiffness of the elastic support on the displacements. Increased stiffness causes the displacement to decrease drastically.

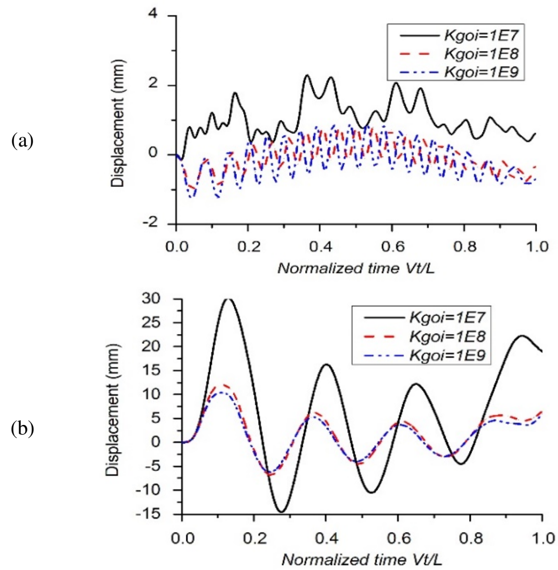

Fig. 8. Displacements for 3 stiffening cases of elastic support. (a) at the center of the beam, (b) at contact point.

V. CONCLUSIONS

The forced vibration of continuous sandwich beams traversed by a moving vehicle has been studied in this paper. Finite element method based on the classical beam theory was used to derive the governing equations of motion of the sandwich beams. The governing equations were solved with the Wilson method. Numerical examples were employed to investigate the effects of vehicle velocity and the stiffness of elastic support on the displacement response of the beam. The result of the numerical examples showed that changes in the stiffness of the elastic support have a significant effect on the displacement response of the beam.

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