Local Instability Analysis for Axial Compressive Steel Columns

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Abstract-This study presents a nonlinear analysis method for steel columns considering local instability under axial compression. The stress-strain relationship of steel is developed while considering the effects of local instability utilizing the energy principles. The second-order effects are considered by stability functions, while the plasticity of steel is captured by a fiber finite element method. Nonlinear equilibrium equations are solved by the generalized displacement control method. As a result, the proposed formulations are effective, accurate, and reliable in predicting the load-carrying capacity and local instability of steel columns.

Keywords-second-order effect; local instability; stability function; steel column; plasticity

I. INTRODUCTION

Steel structures are widely used in several fields (mechanical and civil engineering, aerospace, etc.). The advantages of steel are its high strength, ductility, flexibility, and fast construction, however, instability is a important weak point of steel structures that must be evaluated carefully. Many experimental tests [1-7] were carried out to exam local buckling of steel columns (welded steel columns, high strength steel columns, stainless steel columns, etc.). Examples of local instability of a high strength steel column can be seen in [4]. Authors in [8] studied the local buckling strength of steel columns considering uniformly idealized distributed residual stress. A finite difference method was employed for predicting the plate buckling curves. Boundary conditions for plates were simply supported at loading edges, and free, simply supported, and fixed at the remaining edges. The theoritical results were compared with the experimental tests of 8 square welded steel columns. Author in [9] researched flange local instability of a steel beam with wide-flange shapes. Authors in [10] proposed explicit expressions for the collapse and elastic post-buckling of simply supported plates considering geometric imperfection. In 1977, authors in [11] studied lateral-torsional buckling of local-buckled beams employing a finite element formulation in conjunction with the effective width concept. In this solution, both elastic stiffness and geometric stiffness matrices must be updated at each load step. Authors in [12] developed a finite strip method for the post local-buckled analysis of prismatic thin walled structures under axial compression. Authors in [13] investigated local instability of thin-walled steel box columns. Thin-walled steel box columns were pinned at two ends and were subjected to an eccentrically applied load. In 1997, authors in [5] tested local instability of stub thin-walled circular steel tubes with or without internal restraint considering residual stress and geometric imperfection. More recently, some other studies on local instability for steel members under fire load have been published [6, 14-18].

Local buckling can be easily investigated using plate, shell, or solid elements with the Finite Element Method (FEM). Authors in [19] simulated and proposed the behavior analysis of short concrete-filled steel tube columns using the FEM software ABAQUS. The effects of local buckling of steel tubes were considered by using shell elements. Authors in [20] developed a finite element simulation for normal-strength CFDST members with shear connectors under bending. Authors in [21-23] analyzed several types of structures using the FEM without considering local instability. However, it is not efficient with regard to computational time if these 2D or 3D elements are used. Using line elements, local buckling is implicitly evaluated using design equations of AISC-LRFD specification [24, 25]. Authors in [26-33] studied several works on the nonlinear inelastic behavior of steel frames, but the effects of local instability were not investigated.

In this study, the displacement FEM using one fiber element per member [34-39] is improved to consider local instability. A numerical procedure based on the generalized displacement control method [40] is developed to solve the nonlinear equations of the structural system. The proposed formulations are shown detailly in the following sections.

II. STEEL BEHAVIOR CONSIDERING LOCAL BUCKLING

Local buckling of steel profiles under axial compression is considered by the stress-strain curve as drawn in Figure 2. It is assumed that the whole section will start to buckle at a critical strain ε_{cr}^{C} greater than the yielding strain ε_y . The inelastic

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post-instability response is estimated based on a strain energy method of the component plates as formulated in (1)-(5). The relation between the effective width of the plate b_e and the full width b is:

$$\frac{b_e}{b} \approx \frac{\sigma}{\sigma_y} = \frac{U_B - U_A}{U_B} \quad (1)$$

where σ_y is the yielding stress, ε_y is the yielding strain, U_B and U_A are the strain energy before and after the occurred inelastic local instability. They are formulated as:

$$\begin{aligned} U_B &= \frac{1}{2}\varepsilon_y \sigma_y + \left(\varepsilon_{cr} - \varepsilon_y\right)\sigma_y \\ &= \left(\varepsilon_{cr} - \frac{1}{2}\varepsilon_y\right)\sigma_y \end{aligned} \quad (2)$$

$$U_A = \frac{1}{2}\left(\varepsilon - \varepsilon_{cr}\right)\left(\sigma_y + \sigma\right) \quad (3)$$

in which the critical strain ε_{cr} is calculated from the critical stress σ_{cr} as [41]:

$$\varepsilon_{cr} = \frac{\sigma_{cr}}{E} = \frac{k\pi^2}{12\left(1-v^2\right)\left(b/t\right)^2} \quad (4)$$

where E and v are the elastic modulus and Poisson's ratio of material, b is the considering width of the plate, t is the thickness of the plate, and k is the buckling coefficient given in [42].

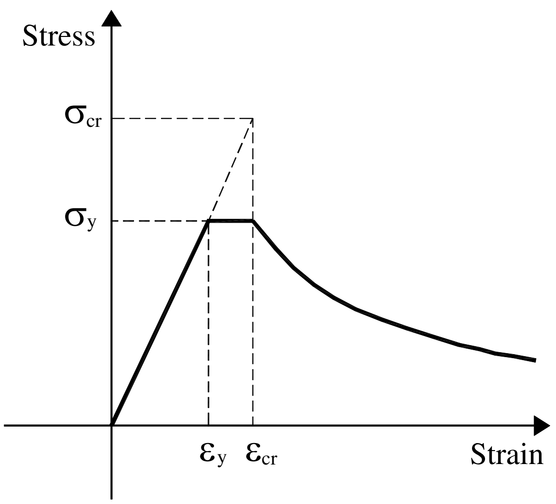

Fig. 1. Assumed stress-strain curve for local buckling of steel.

After substituting (2) and (3) into (1), the inelastic post-local instability behavior can be estimated by:

$$\frac{\sigma}{\sigma_y} = \frac{1}{1+\alpha\left(\dfrac{1-\dfrac{\varepsilon_{cr}}{\varepsilon}}{1-\dfrac{\varepsilon_y}{2\varepsilon}}\right)} \quad (5)$$

where α is a adjusting parameter of the slope of the post-instability behavior.

III. FORMULAS FOR THE NONLINEAR COLUMN ELEMENT

A. *Second-Order and Shear Deformation Effects*

The stability functions proposed in [43] are employed to simulate the second-order effect by using only one element per member. From [34, 35, 38, 39], the incremental equilibrium equation of a column element, considering both shear deformation and the second-order effects is written as:

$$\begin{Bmatrix} \Delta P \\ \Delta M_{yA} \\ \Delta M_{yB} \\ \Delta M_{zA} \\ \Delta M_{zB} \\ \Delta T \end{Bmatrix} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & C_{1y} & C_{2y} & 0 & 0 & 0 \\ 0 & C_{2y} & C_{1y} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{1z} & C_{2z} & 0 \\ 0 & 0 & 0 & C_{2z} & C_{1z} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{GJ}{L} \end{bmatrix} \begin{Bmatrix} \Delta\delta \\ \Delta\theta_{yA} \\ \Delta\theta_{yB} \\ \Delta\theta_{zA} \\ \Delta\theta_{zB} \\ \Delta\phi \end{Bmatrix} \quad (6)$$

where ΔP, ΔT, ΔM_{yA}, ΔM_{yB}, ΔM_{zA}, and ΔM_{zB} are the axial force, torsion and bending moments respectively, $\Delta\delta$, $\Delta\phi$, $\Delta\theta_{yA}$, $\Delta\theta_{yB}$, $\Delta\theta_{zA}$, and $\Delta\theta_{zB}$ are the axial shortening and rotations respectively, A is the sectional area, J is the moment of inertia around the z axis, L is the element length, G is the shear modulus of steel, and C_{1y}, C_{2y}, C_{1z}, and C_{2z} are respectively the flexural stiffness coefficients considering the shear deformation, estimated as:

$$C_{1y} = \frac{k_{1y}^2 - k_{2y}^2 + k_{1y}A_{sz}GL}{2k_{1y} + 2k_{2y} + A_{sz}GL} \quad (7)$$

$$C_{2y} = \frac{-k_{1y}^2 + k_{2y}^2 + k_{2y}A_{sz}GL}{2k_{1y} + 2k_{2y} + A_{sz}GL} \quad (8)$$

$$C_{1z} = \frac{k_{1z}^2 - k_{2z}^2 + k_{1z}A_{sy}GL}{2k_{1z} + 2k_{2z} + A_{sy}GL} \quad (9)$$

$$C_{2z} = \frac{-k_{1z}^2 + k_{2z}^2 + k_{2y}A_{sy}GL}{2k_{1z} + 2k_{2z} + A_{sy}GL} \quad (10)$$

where $k_{1n} = S_{1n}\left(EI_n / L\right)$ and $k_{2n} = S_{2n}\left(EI_n / L\right)$. S_{1n} and S_{2n} are the stability functions with the axis of n $\left(n = y, z\right)$ and are written as:

$$S_{1n}=\begin{cases}\dfrac{h_nL\sin(h_nL)-(h_nL)^2\cos(h_nL)}{2-2\cos(h_nL)-h_nL\sin(h_nL)} & \text{if } P<0\\ \dfrac{(h_nL)^2\cosh(h_nL)-h_nL\sinh(h_nL)}{2-2\cosh(h_nL)+h_nL\sinh(h_nL)} & \text{if } P>0\end{cases} \quad (11)$$

$$S_{2n}=\begin{cases}\dfrac{(h_nL)^2-h_nL\sin(h_nL)}{2-2\cos(h_nL)-h_nL\sin(h_nL)} & \text{if } P<0\\ \dfrac{h_nL\sin(h_nL)-(h_nL)^2}{2-2\cosh(h_nL)+h_nL\sinh(h_nL)} & \text{if } P>0\end{cases} \quad (12)$$

which $h_n^2=|P|/EI_n$. EA and EI_n are the axial and flexural stiffnesses of the column and are calculated by:

$$EA=\sum_{j=1}^{p} w_j\left(\sum_{i=1}^{m} E_iA_i\right)_j \quad (13)$$

$$EI_y=\sum_{j=1}^{p} w_j\left[\sum_{i=1}^{m} E_i\left(A_iz_i^2+I_{yi}\right)\right]_j \quad (14)$$

$$EI_z=\sum_{j=1}^{p} w_j\left[\sum_{i=1}^{m} E_i\left(A_iy_i^2+I_{zi}\right)\right]_j \quad (15)$$

where p is the number of meshed sections per element, m is the number of fibres divided on the meshed sections, w_j is the weight parameters for Lobatto quadrature at the j^{th} section [44], E_i and A_i are the elastic modulus of the material and the area of i^{th} fiber respectively, I_{yi} and I_{zi} are the y-axis and z-axis moments of inertia of the i^{th} fiber around its centroid, y_i and z_i are the coordinates of the i^{th} fiber to the central bending axis of the section as illustrated in Figure 2.

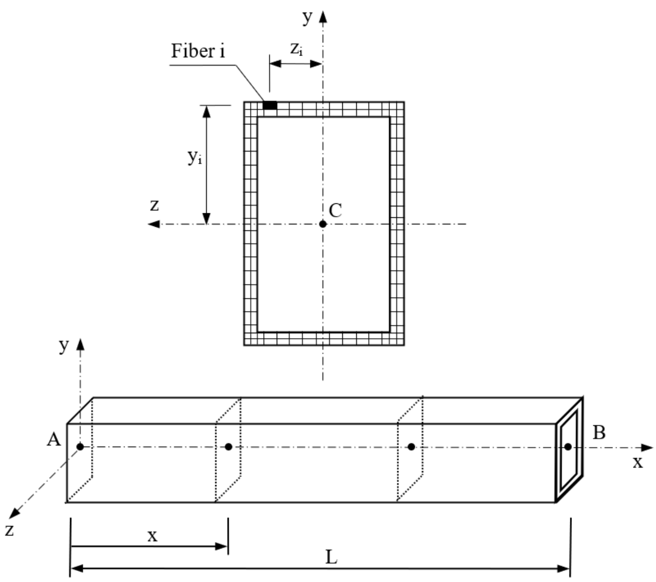

Fig. 2. Meshing of the steel column.

B. *Effects of Plasticity*

To predict the plasticity along the member, a fiber column model as plotted in Figure 2 is used. Figure 3 shows the residual stress pattern shape [45] used in this study. The section deformations are estimated by three strain resultants: the axial strain ε and curvatures χ_z and χ_y. The force resultants are the axial force N and the flexural moments M_z and M_y. The sectional forces and deformations are calculated as:

$$\text{Sectional force } \{Q\}=\begin{bmatrix}N & M_y & M_z\end{bmatrix}^T \quad (16)$$

$$\text{Sectional deformation } \{q\}=\begin{bmatrix}\varepsilon & \chi_y & \chi_z\end{bmatrix}^T \quad (17)$$

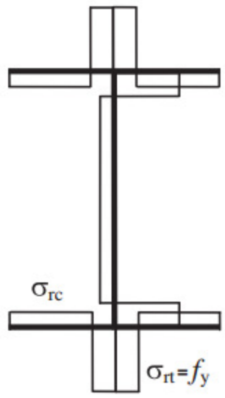

Fig. 3. Assumed residual stress pattern for the steel column.

IV. VERIFICATION

In [46], two experimental tests were carried out for I-sections with strengthening stiffeners at column bases. The elastic modulus is 200000MPa and the yield stress is 265MPa. The ratio of the width per the thickness (*b/t*) of specimens changed from 20 to 25. The initial residual stresses measured in [46] were 17% to 19% of the yield stress as listed in Table I, along with the column geometries. The load ratio-axial shortening curves captured by the proposed formulation are compared with ABAQUS in Figures 4 and 5. It can be seen that the proposed curve including local buckling nearly matches with the result of ABAQUS (Figure 4). Table II presents a comparison of the ultimate compressive load estimated by experimetal tests, ABAQUS, and the proposed formulation (LB: Local Buckling). The ultimate load estimated by the proposed formulation is slightly higher than the results of experimental tests and ABAQUS. There are big differences between considering local buckling or not in the analysis procedure. The clear differences occur due to the plastic elastic regime of the steel fibers.

TABLE I. GEOMETRIES OF SPECIMENS TESTED IN [46]

Specimen	Geometric characteristics					
	***b* (mm)**	***t* (mm)**	***b/t***	***A* (mm^2)**	***L* (mm)**	**σ_{rc}/σ_y**
HI1	60	3	20	1062	900	0.18
HI2	75	3	25	1332	900	0.19

TABLE II. COMPARISON OF ULTIMATE COMPRESSION LOAD

Specimen	P_{Test} (kN) [46]	P_{Ana}/P_{Test}		
		ABAQUS	Proposed (with LB)	Proposed (without LB)
HI1	260	0.949	1.011	1.090
HI2	250	1.075	1.205	1.427

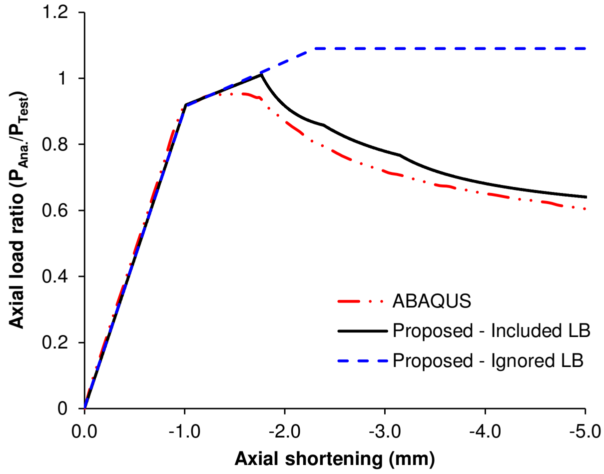

Fig. 4. Load ratio-axial shortening curve of HI1 column.

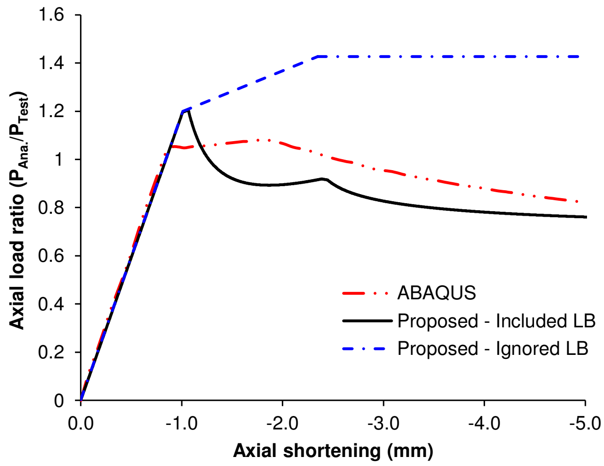

Fig. 5. Load ratio-axial shortening curve of HI2 column.

V. CONCLUSION

In this paper, the finite element analysis formulation for the second-order inelastic steel columns considering local instability was developed successfully. By using one element per member, the proposed formulation reduced computational time and computer resources. The results are in good accordance with the results of ABAQUS and the experimental tests. The proposed formulation can be applied in the development of practical engineering software.

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REFERENCES

[1] B. Uy, "Local and post-local buckling of concrete filled steel welded box columns," *Journal of Constructional Steel Research*, vol. 47, no. 1, pp. 47–72, Aug. 1998, https://doi.org/10.1016/S0143-974X(98)80102-8.

[2] L. Gardner and M. Theofanous, "Discrete and continuous treatment of local buckling in stainless steel elements," *Journal of Constructional Steel Research*, vol. 64, no. 11, pp. 1207–1216, Nov. 2008, https://doi.org/10.1016/j.jcsr.2008.07.003.

[3] G. Shi, K. Xu, H. Ban, and C. Lin, "Local buckling behavior of welded stub columns with normal and high strength steels," *Journal of Constructional Steel Research*, vol. 119, pp. 144–153, Mar. 2016, https://doi.org/10.1016/j.jcsr.2015.12.020.

[4] A. Sharhan, W. Wang, X. Li, and H. Al-azzani, "Steady and transient state tests on local buckling of high strength Q690 steel stub columns," *Thin-Walled Structures*, vol. 167, Oct. 2021, Art. no. 108214, https://doi.org/10.1016/j.tws.2021.108214.

[5] M. D. O'Shea and R. Q. Bridge, "Local buckling of thin-walled circular steel sections with or without internal restraint," *Journal of Constructional Steel Research*, vol. 41, no. 2, pp. 137–157, Feb. 1997, https://doi.org/10.1016/S0143-974X(97)80891-7.

[6] K.-C. Yang, S.-J. Chen, C.-C. Lin, and H.-H. Lee, "Experimental study on local buckling of fire-resisting steel columns under fire load," *Journal of Constructional Steel Research*, vol. 61, no. 4, pp. 553–565, Apr. 2005, https://doi.org/10.1016/j.jcsr.2004.07.001.

[7] G. Shi, W. Zhou, and C. Lin, "Experimental Investigation on the Local Buckling Behavior of 960 MPa High Strength Steel Welded Section Stub Columns," *Advances in Structural Engineering*, vol. 18, no. 3, pp. 423–437, Mar. 2015, https://doi.org/10.1260/1369-4332.18.3.423.

[8] F. Nishino and L. Tall, "Residual Stress and Local Buckling Strength of Steel Columns," *Proceedings of the Japan Society of Civil Engineers*, vol. 1969, no. 172, pp. 79–96, 1969, https://doi.org/10.2208/jscej1969.1969.172_79.

[9] M. G. Lay, "Flange Local Buckling in Wide-Flange Shapes," *Journal of the Structural Division*, vol. 91, no. 6, pp. 95–116, Dec. 1965, https://doi.org/10.1061/JSDEAG.0001371.

[10] R. G. Dawson and A. C. Walker, "Post-Buckling of Geometrically Imperfect Plates," *Journal of the Structural Division*, vol. 98, no. 1, pp. 75–94, Jan. 1972, https://doi.org/10.1061/JSDEAG.0003145.

[11] S. T. Wang, M. I. Yost, and Y. L. Tien, "Lateral buckling of locally buckled beams using finite element techniques," *Computers & Structures*, vol. 7, no. 3, pp. 469–475, Jun. 1977, https://doi.org/10.1016/0045-7949(77)90084-0.

[12] T. R. Graves Smith and S. Sridharan, "A finite strip method for the post-locally-buckled analysis of plate structures," *International Journal of Mechanical Sciences*, vol. 20, no. 12, pp. 833–842, Jan. 1978, https://doi.org/10.1016/0020-7403(78)90009-7.

[13] J. Y. Richard Liew, N. E. Shanmugam, and S. L. Lee, "Local buckling of thin-walled steel box columns," *Thin-Walled Structures*, vol. 8, no. 2, pp. 119–145, Jan. 1989, https://doi.org/10.1016/0263-8231(89)90039-6.

[14] M. Knobloch and M. Fontana, "Strain-based approach to local buckling of steel sections subjected to fire," *Journal of Constructional Steel Research*, vol. 62, no. 1, pp. 44–67, Jan. 2006, https://doi.org/10.1016/j.jcsr.2005.04.007.

[15] C. Maraveas, "Local Buckling of Steel Members Under Fire Conditions: A Review," *Fire Technology*, vol. 55, no. 1, pp. 51–80, Jan. 2019, https://doi.org/10.1007/s10694-018-0768-1.

[16] Z. Xing, M. Kucukler, and L. Gardner, "Local buckling of stainless steel plates in fire," *Thin-Walled Structures*, vol. 148, Mar. 2020, Art. no. 106570, https://doi.org/10.1016/j.tws.2019.106570.

[17] W. Wang, X. Li, and H. Al-azzani, "Experimental study on local buckling of high-strength Q960 steel columns at elevated temperatures," *Journal of Constructional Steel Research*, vol. 183, Aug. 2021, Art. no. 106716, https://doi.org/10.1016/j.jcsr.2021.106716.

[18] Z. Xing, M. Kucukler, and L. Gardner, "Local buckling of stainless steel I-sections in fire: Finite element modelling and design," *Thin-Walled Structures*, vol. 161, Apr. 2021, Art. no. 107486, https://doi.org/10.1016/j.tws.2021.107486.

[19] P. C. Nguyen, D. D. Pham, T. T. Tran, and T. Nghia-Nguyen, "Modified Numerical Modeling of Axially Loaded Concrete-Filled Steel Circular-Tube Columns," *Engineering, Technology & Applied Science Research*, vol. 11, no. 3, pp. 7094–7099, Jun. 2021, https://doi.org/10.48084/etasr.4157.

[20] S.-E. Kim *et al.*, "Finite element simulation of normal – Strength CFDST members with shear connectors under bending loading," *Engineering Structures*, vol. 238, Jul. 2021, Art. no. 112011, https://doi.org/10.1016/j.engstruct.2021.112011.

[21] T.-T. Tran, M. Hussan, D. Kim, and P.-C. Nguyen, "Distributed plasticity approach for the nonlinear structural assessment of offshore wind turbine," *International Journal of Naval Architecture and Ocean Engineering*, vol. 12, pp. 743–754, Jan. 2020, https://doi.org/10.1016/j.ijnaoe.2020.09.003.

[22] T.-T. Tran, P.-C. Nguyen, G. So, and D. Kim, "Seismic behavior of steel cabinets considering nonlinear connections and site-response effects," *Steel and Composite Structures*, vol. 36, no. 1, pp. 17–29, Jul. 2020, https://doi.org/10.12989/scs.2020.36.1.017.

[23] T.-T. Tran *et al.*, "Probabilistic Seismic Demand Model and Seismic Fragility Analysis of NPP Equipment Subjected to High- and Low-Frequency Earthquakes," *Nuclear Science and Engineering*, vol. 195, no. 12, pp. 1327–1346, Dec. 2021, https://doi.org/10.1080/00295639.2021.1920796.

[24] S.-E. Kim, J. Lee, and J.-S. Park, "3-D second-order plastic-hinge analysis accounting for local buckling," *Engineering Structures*, vol. 25, no. 1, pp. 81–90, Jan. 2003, https://doi.org/10.1016/S0141-0296(02)00122-0.

[25] S.-E. Kim and J. Lee, "Improved refined plastic-hinge analysis accounting for local buckling," *Engineering Structures*, vol. 23, no. 8, pp. 1031–1042, Aug. 2001, https://doi.org/10.1016/S0141-0296(00)00105-X.

[26] C. Ngo-Huu, P.-C. Nguyen, and S.-E. Kim, "Second-order plastic-hinge analysis of space semi-rigid steel frames," *Thin-Walled Structures*, vol. 60, pp. 98–104, Nov. 2012, https://doi.org/10.1016/j.tws.2012.06.019.

[27] P.-C. Nguyen and S.-E. Kim, "Nonlinear elastic dynamic analysis of space steel frames with semi-rigid connections," *Journal of Constructional Steel Research*, vol. 84, pp. 72–81, May 2013, https://doi.org/10.1016/j.jcsr.2013.02.004.

[28] P.-C. Nguyen, N. T. N. Doan, C. Ngo-Huu, and S.-E. Kim, "Nonlinear inelastic response history analysis of steel frame structures using plastic-zone method," *Thin-Walled Structures*, vol. 85, pp. 220–233, Dec. 2014, https://doi.org/10.1016/j.tws.2014.09.002.

[29] P.-C. Nguyen and S.-E. Kim, "Nonlinear inelastic time-history analysis of three-dimensional semi-rigid steel frames," *Journal of Constructional Steel Research*, vol. 101, pp. 192–206, Oct. 2014, https://doi.org/10.1016/j.jcsr.2014.05.009.

[30] P.-C. Nguyen and S.-E. Kim, "Distributed plasticity approach for time-history analysis of steel frames including nonlinear connections," *Journal of Constructional Steel Research*, vol. 100, pp. 36–49, Sep. 2014, https://doi.org/10.1016/j.jcsr.2014.04.012.

[31] P.-C. Nguyen and S.-E. Kim, "Advanced analysis for planar steel frames with semi-rigid connections using plastic-zone method," *Steel and Composite Structures*, vol. 21, no. 5, pp. 1121–1144, Jan. 2016.

[32] P.-C. Nguyen and S.-E. Kim, "Investigating effects of various base restraints on the nonlinear inelastic static and seismic responses of steel frames," *International Journal of Non-Linear Mechanics*, vol. 89, pp. 151–167, Mar. 2017, https://doi.org/10.1016/j.ijnonlinmec.2016.12.011.

[33] P.-C. Nguyen and S.-E. Kim, "A new improved fiber plastic hinge method accounting for lateral-torsional buckling of 3D steel frames," *Thin-Walled Structures*, vol. 127, pp. 666–675, Jun. 2018, https://doi.org/10.1016/j.tws.2017.12.031.

[34] P.-C. Nguyen and S.-E. Kim, "An advanced analysis method for three-dimensional steel frames with semi-rigid connections," *Finite Elements in Analysis and Design*, vol. 80, pp. 23–32, Mar. 2014, https://doi.org/10.1016/j.finel.2013.11.004.

[35] P.-C. Nguyen and S.-E. Kim, "Second-order spread-of-plasticity approach for nonlinear time-history analysis of space semi-rigid steel frames," *Finite Elements in Analysis and Design*, vol. 105, pp. 1–15, Nov. 2015, https://doi.org/10.1016/j.finel.2015.06.006.

[36] P. C. Nguyen, "Nonlinear Inelastic Earthquake Analysis of 2D Steel Frames," *Engineering, Technology & Applied Science Research*, vol. 10, no. 6, pp. 6393–6398, Dec. 2020, https://doi.org/10.48084/etasr.3855.

[37] P. C. Nguyen, B. Le-Van, and S. D. T. V. Thanh, "Nonlinear Inelastic Analysis of 2D Steel Frames : An Improvement of the Plastic Hinge Method," *Engineering, Technology & Applied Science Research*, vol. 10, no. 4, pp. 5974–5978, Aug. 2020, https://doi.org/10.48084/etasr.3600.

[38] P.-C. Nguyen and T. D. Tran, "Impacts of residual stress and shear deformation on 2D steel frames using fiber plastic hinge element: nonlinear behavior and strength," *SN Applied Sciences*, vol. 3, no. 7, Jul. 2021, Art. no. 686, https://doi.org/10.1007/s42452-021-04638-w.

[39] P.-C. Nguyen, T.-T. Tran, and T. Nghia-Nguyen, "Nonlinear time-history earthquake analysis for steel frames," *Heliyon*, vol. 7, no. 8, Aug. 2021, Art. no. e06832, https://doi.org/10.1016/j.heliyon.2021.e06832.

[40] Y.-B. Yang and M.-S. Shieh, "Solution method for nonlinear problems with multiple critical points," *AIAA Journal*, vol. 28, no. 12, pp. 2110–2116, 1990, https://doi.org/10.2514/3.10529.

[41] G. H. Bryan, "On the Stability of a Plane Plate under Thrusts in its own Plane, with Applications to the 'Buckling' of the Sides of a Ship," *Proceedings of the London Mathematical Society*, vol. s1-22, no. 1, pp. 54–67, Nov. 1890, https://doi.org/10.1112/plms/s1-22.1.54.

[42] J. H. Lee, "Local Buckling Behaviour and Design of Cold-formed Steel Compression Members at Elevated Temperatures," Ph.D. dissertation, Queensland University of Technology, Brisbane, QLD, Australia, 2004.

[43] W.-F. Chen, *Structural Stability: Theory and Implementation*, Illustrated edition. Englewood Cliffs, NJ, USA: Prentice Hall, 1987.

[44] H. H. Michels, "Abscissas and weight coefficients for Lobatto quadrature," *Mathematics of Computation*, vol. 17, no. 83, pp. 237–244, 1963, https://doi.org/10.1090/S0025-5718-1963-0158540-4.

[45] S. Kitipornchai and A. D. Wong-Chung, "Inelastic Buckling of Welded Monosymmetric I-Beams," *Journal of Structural Engineering*, vol. 113, no. 4, pp. 740–756, Apr. 1987, https://doi.org/10.1061/(ASCE)0733-9445(1987)113:4(740).

[46] B. Uy, "Local and Postlocal Buckling of Fabricated Steel and Composite Cross Sections," *Journal of Structural Engineering*, vol. 127, no. 6, pp. 666–677, Jun. 2001, https://doi.org/10.1061/(ASCE)0733-9445(2001)127:6(666).