A Novel Two-Parameter Elastic Foundation Model for the Bending Analysis of Functionally Graded Beams

Aizhan Nurgoziyeva
Laboratory of Applied Mechanics and Robotics, Buketov Karaganda National Research University, Karaganda, Kazakhstan
aizhanzhanabai@gmail.com

Sungat Akhazhanov
Laboratory of Applied Mechanics and Robotics, Buketov Karaganda National Research University, Karaganda, Kazakhstan
akhazhanov.s@karnu-buketov.edu.kz (corresponding author)

Meiirrassul Bekov
Laboratory of Applied Mechanics and Robotics, Buketov Karaganda National Research University, Karaganda, Kazakhstan
chinkoplusmanko@gmail.com

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ABSTRACT

This study presents an analytical bending analysis of Functionally Graded (FG) beams resting on a newly developed two-parameter elastic foundation. Unlike classical foundation models, such as the Winkler, Pasternak, and Vlasov formulations, the proposed foundation model incorporates two independent parameters, enabling a more flexible and physically realistic description of the coupled normal and shear interactions between the beam and the supporting medium. The material properties of the FG beam are assumed to vary continuously through the thickness according to a power-law distribution. The governing differential equations were derived within the framework of the beam theory and solved analytically. Closed-form solutions for transverse deflections, bending moments, and stress resultants are obtained, enabling a detailed assessment of the bending response of FG beams. A parametric study is performed to investigate the effects of material gradation, beam slenderness, and foundation parameters on displacement and stress distributions. The results demonstrate that the proposed foundation parameters significantly influence the bending behavior of FG beams, particularly for thick beams and higher material gradation indices, resulting in noticeable deviations from predictions based on classical elastic foundation models. Comparisons with available solutions from the literature for limiting cases, including the homogeneous metallic beam, show excellent agreement and confirm the accuracy and reliability of the present formulation. The smooth convergence to classical solutions verifies the mathematical consistency and physical validity of the proposed approach, which provides an effective framework for the advanced analysis of FG beams on elastic foundations.

Keywords-functionally graded beam; two-parameter elastic foundation; analytical bending analysis; axial displacement; transverse displacement; transverse shear stress; axial stress

I. INTRODUCTION

Elastic foundation models are widely used to describe the interaction between structural elements and deformable supporting media in many engineering applications, including soil–structure interaction, railway tracks, pavement systems, layered composites, and foundation-supported beams. Among the earliest and most commonly employed representations is the Winkler foundation model, which idealizes the supporting medium as a set of independent linear springs acting normal to the structural element [1]. Despite its simplicity and computational efficiency, the Winkler model neglects shear interaction and continuity within the foundation, which may lead to inaccurate predictions of structural response in practical engineering problems.

To overcome these limitations, several refined elastic foundation models have been proposed. The Pasternak foundation introduces an additional shear layer that accounts for the interaction between adjacent foundation points, resulting in a two-parameter model that significantly improves the accuracy of beam and plate analyses [2, 3]. Further developments include the Vlasov-Leont'ev foundation model, which treats the foundation as a continuous elastic medium and captures long-range interaction effects [4]. Although these advanced models provide a more realistic description of foundation behavior, they generally increase mathematical complexity and restrict the possibility of deriving closed-form analytical solutions, particularly for Functionally Graded (FG) structures.

Functionally Graded Materials (FGMs) constitute a class of advanced composites characterized by a continuous variation of material properties, typically along the thickness direction. This smooth gradation avoids abrupt material interfaces typical of layered composites, thereby reducing stress concentrations and improving the overall structural integrity [6]. Owing to these advantages, FGMs have been increasingly employed in aerospace, mechanical, civil, and thermal engineering applications [5, 7]. Beams made of FGMs serve as significant structural elements in many of these systems, which has motivated extensive research on their bending, vibration, and stability behavior.

The mechanical response of FG beams has been investigated using various beam theories, ranging from the classical Euler-Bernoulli and Timoshenko theories to refined and higher-order shear deformation models [8-11]. For slender beams, the Euler-Bernoulli beam theory remains particularly attractive due to its simplicity and suitability for analytical modeling. The bending behavior of FG beams resting on elastic foundations has been examined using Winkler and Winkler-Pasternak models through analytical, semi-analytical, and numerical approaches [12-17]. It has been demonstrated that both the material gradation profile and the foundation stiffness parameters play a crucial role in governing beam deflections and internal force distributions.

Recent studies have further extended FG beam models to incorporate nonlinear foundations, viscoelastic effects, porosity, thermal loading, and size-dependent behavior based on nonlocal or strain-gradient elasticity theories [19, 21, 22]. While these approaches enhance modeling accuracy, they often rely on complex formulations or numerical schemes, which may obscure the physical interpretation of foundation parameters and limit their applicability as benchmark solutions. Moreover, most existing works are still based on classical foundation formulations – primarily Winkler, Pasternak, or Vlasov models-which may not offer sufficient flexibility to represent diverse beam-foundation interaction mechanisms. Two-parameter elastic foundation models, particularly the Winkler–Pasternak formulation and its extensions, have been employed to analyze FG beams [12–18, 20–22]. These models significantly improve the representation of shear interaction within the foundation compared to the classical Winkler approach.

Despite the extensive literature on FG beams supported by elastic foundations, there is a lack of analytically tractable models that introduce alternative two-parameter elastic foundation formulations beyond the classical Pasternak concept. In particular, relatively few studies have focused on developing new foundation models that achieve a balance between physical realism and analytical simplicity within the framework of classical beam theories. This limitation restricts the availability of closed-form solutions that can serve as reliable reference results for validating numerical methods and higher-order theories.

Thus, the present study proposes a novel two-parameter elastic foundation model for the analytical bending analysis of FG beams. The formulation is developed within the Euler-Bernoulli beam theory framework, assuming a power-law distribution of material properties through the beam thickness. The influence of the elastic foundation is incorporated through parameters associated with the flexural response of the beam-foundation system. The governing differential equation for the bending of FG beams resting on the proposed two-parameter elastic foundation is derived analytically. A comprehensive parametric investigation is conducted to examine the effects of the foundation parameters, material gradation index, and geometric characteristics on the bending response of FG beams. Comparisons with available results from the literature for selected limiting cases are performed to verify the accuracy and reliability of the proposed model. The developed analytical framework provides a useful and efficient tool for advanced modeling and benchmark analysis of FG beams resting on elastic foundations.

II. THEORETICAL FORMULATION

In the present study, an FG rectangular beam of uniform thickness h_0, length l, and rectangular cross-section $b \times h_0$ is considered. The beam is supported by a two-parameter elastic foundation of thickness h, where $\overline{E}$ denotes the elastic modulus of the foundation material, as illustrated in Figure 1.

The following assumptions are adopted in the present study:

- The beam is modeled according to the Euler–Bernoulli beam theory.
- The material properties vary continuously through the thickness according to a power-law distribution.
- The Poisson's ratio is assumed to be constant.
- The elastic foundation is modeled as a two-parameter continuum.
- The foundation is considered a semi-infinite elastic medium.
- The contact between the beam and the foundation is perfectly bonded.
- The analysis is performed within the framework of linear elasticity.

The coordinate axes x_1 and x_3 are oriented along the longitudinal and thickness directions of the beam, respectively. The FG beam is considered in the domain:

$$-\frac{l}{2} \leq x_1 \leq \frac{l}{2}, -\frac{h_0}{2} \leq x_3 \leq \frac{h_0}{2}$$

The elastic foundation is described in a coordinate system:

$$-\infty \leq x_1 \leq \infty, 0 \leq x_3 \leq \infty$$

The contact between the beam and the elastic foundation is assumed to be perfectly bonded, ensuring the continuity of displacements and stresses at the interface.

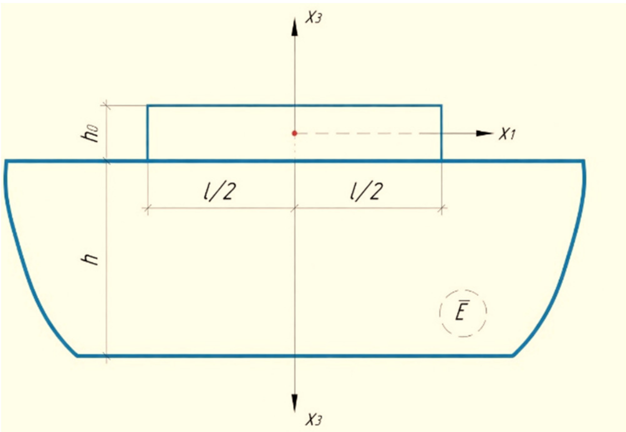

Fig. 1. FG beam resting on the proposed two-parameter elastic foundation.

The FG beam consists of a ceramic phase (alumina) and a metallic phase (aluminum), whose material properties vary smoothly through the thickness direction. This material combination is widely used in FGMs due to the complementary properties of its constituents. Alumina provides high stiffness, strength, and thermal resistance, while aluminum offers good ductility, toughness, and lower density. Such a combination enables a smooth transition of material properties through the thickness, resulting in improved structural performance and reduced stress concentrations. The variation of the Young's modulus across the beam thickness is described by a power-law function, which reflects the continuous transition between the constituent materials:

$$E(z) = E_m + (E_c - E_m)\left(\frac{1}{2} + z\right)^p \quad (1)$$

where E_m and E_c denote the Young's moduli at the top and bottom surfaces of the beam, respectively, while $E(z)$ represents the thickness-dependent Young's modulus of the FGMs. The parameter p is the power-law index governing the material gradation. The Poisson's ratio ν is assumed to remain constant throughout the beam thickness and is used in the subsequent constitutive relations.

Based on the Euler-Bernoulli beam theory [23], the displacement field adopted in the present study is expressed as:

$$U_3^0 = W_0(x_1), U_1^0 = -h_0 \cdot \phi_0(z)\frac{dW_0(x_1)}{dx_1}, \quad (2)$$

$$\phi_0(z) = -C_0 + z,\ z = \frac{x_3}{h_0}$$

where U_1^0 and U_3^0 denote the displacements of the FG beam along the coordinate axes x_1 and x_3, respectively, $\phi_0(z)$ denotes the function describing the distribution of shear displacements in the x_1, x_3 - plane, $W_0(x_1)$ represents the transverse deflection of the beam, C_0 is an arbitrary constant, and z is the dimensionless thickness coordinate.

Taking into account Hooke's constitutive relations, $\sigma_1^0 = E(z)$, $\varepsilon_1 = E(z)\frac{\partial U_1^0}{\partial x_1}$ and the equilibrium equations, $\frac{\partial \sigma_1^0}{\partial x_1} + \frac{\partial \tau_{13}^0}{\partial x_3} = 0$, $\frac{\partial \tau_{13}^0}{\partial x_1} + \frac{\partial \sigma_3^0}{\partial x_3} = 0$, the stress components of the beam can be expressed as:

$$\sigma_1^0 = -E(z)h_0\phi_0(z)\frac{d^2W_0(x_1)}{dx_1^2}$$

$$\tau_{13}^0 = E(z)h_0^2\psi_0(z)\frac{d^3W_0(x_1)}{dx_1^3}$$

$$\sigma_3^0 = E(z)h_0^3\delta_0(z)\frac{d^4W_0(x_1)}{dx_1^4} \quad (3)$$

$$\psi_0(z) = A_0 - C_0 z + \frac{z^2}{2}$$

$$\delta_0(z) = B_0 - A_0 z + C_0\frac{z^2}{2} - \frac{z^3}{6}$$

where σ_1^0, σ_3^0 denote the normal stress components acting on the FG beam along the x_1, x_3 - coordinate directions, respectively, evaluated at the reference surface of the beam, τ_{13}^0 represents the tangential stress of the FG beam, $\psi_0(z)$ and $\delta_0(z)$ are functions describing the distribution of transverse shear stress τ_{13}^0 and normal stresses across the beam thickness, respectively, and A_0, B_0, C_0 are arbitrary constants.

The displacement and stress components of the proposed novel two-parameter elastic foundation are determined using [24]:

$$U_1 = -h\phi(z_0)\frac{dW(x_1)}{dx_1}$$

$$U_3 = f(z_0)W(x_1)$$

$$\phi(z_0) = \frac{(1+\nu)}{(1-\nu)}\delta'(z_0)$$

$$f(z_0) = \delta''(z_0) - \frac{2}{(1-\nu)}k^2\delta(z_0) \quad (4)$$

$$\delta(z_0) = \left\{\frac{1}{12(1+\nu)^2}[(1-\nu)\beta_0 + 2k\nu\alpha_0] + \frac{k}{12(1+\nu)}[k\alpha_0 - \beta_0]z_0\right\}e^{-kz_0}$$

$$z_0 = \frac{x_3}{h}$$

$$k^2 = \bar{k}^2 \cdot h^2$$

where U_1 and U_3 represent the displacements of the elastic foundation along the coordinate axes x_1 and x_3, respectively, $W(x_1)$ denotes the deflection function of the elastic foundation, ν denotes the Poisson's ratio of the elastic foundation material, z_0 is the dimensionless transverse coordinate, $\phi(z_0)$ and $f(z_0)$ are functions describing the distribution of the displacement components U_1 and U_3, respectively, $\delta(z_0)$ denotes the primary displacement distribution function through the thickness, and k is a parameter characterizing the deformed state, which depends on the applied boundary conditions, and z_0 denotes the dimensionless transverse coordinate.

$$\sigma_1 = -\frac{\overline{E}h}{12}\cdot\psi'(z_0)\frac{d^2W(x_1)}{dx_1^2}$$
$$\tau_{13} = \frac{\overline{E}h^2}{12}\cdot\psi(z_0)\frac{d^3W(x_1)}{dx_1^3}$$
$$\sigma_3 = \frac{\overline{E}h^3}{12}\cdot\alpha(z_0)\frac{d^4W(x_1)}{dx_1^4} \quad (5)$$
$$\psi(z_0) = 12\left[\delta(z_0) + \frac{\nu}{k^2}\delta''(z_0)\right]$$
$$\alpha(z_0) = \frac{12}{k^4}[\delta'''(z_0) - (2+\nu)k^2\delta'(z_0)]$$

where σ_1 and σ_3 denote the normal stresses of the elastic foundation along the coordinate axes x_1 and x_3, respectively, τ_{13} is the shear stress of the elastic foundation, and $\psi'(z_0)$, $\psi(z_0)$, $\alpha(z_0)$ are functions describing the distribution of the corresponding stress components across the thickness of the elastic foundation.

Using (2)-(5), together with the contact and boundary conditions, the arbitrary constants (A_0, B_0, C_0) are determined:

$$C_0 = \frac{1}{12}\frac{\overline{E}h^2}{E_m h_0^2}\beta_0$$
$$A_0 = -\frac{1}{8} + \frac{1}{24}\frac{\overline{E}h^2}{E_m h_0^2}\beta_0 \quad (6)$$
$$B_0 = \frac{1}{24} - \frac{1}{32}\frac{\overline{E}h^2}{E_m h_0^2}\beta_0 - \frac{1}{12}\frac{\overline{E}h^3}{E_m h_0^3}\alpha_0$$

where α_0 and β_0 denote the values of the elastic foundation stress distribution functions at the contact point.

By taking into account the distribution functions given in (4) and applying the condition that the beam deflection is equal to the foundation deflection at the contact point, the parameters α_0 and β_0 are determined as:

$$\alpha_0 = -\frac{6(1-\nu^2)}{k^3}P_0,\ \beta_0 = -\frac{12(1-\nu^2)}{k^2}P_1$$
$$P_0 = \frac{2\cdot n + \frac{h_0}{h}k(1-\nu)}{2n-(1-\nu)^2},\ P_1 = \frac{k\frac{h_0}{h} + (1-\nu)}{2n-(1-\nu)^2} \quad (7)$$
$$c = \frac{6(1-\nu^2)}{k},\ n = 2 + \frac{(1-\nu^2)}{k}\frac{\overline{E}h}{E_m h_0}$$

where P_0, P_1, c, and n denote the main parameters.

The boundary conditions for the FG beam at $z = \frac{1}{2}$ are expressed as:

$$\sigma_3^0 = q:\ E\left(\frac{1}{2}\right)h_0^3\delta_0\left(\frac{1}{2}\right)\frac{d^4W_0(x_1)}{dx_1^4} = q(x_1)$$

By substituting these boundary conditions into (3), the governing differential equation for the bending of the FG beam resting on a two-parameter elastic foundation is obtained:

$$\gamma\frac{d^4W_0(x_1)}{dx_1^4} = \frac{q(x_1)}{\left(E_m + (E_c - E_m)\left(\frac{1}{4}\right)^p\right)J}$$
$$\gamma = 1 + \frac{6(1-\nu^2)\cdot P_1}{k^2}\frac{\overline{E}h^2}{E_m h_0^2} + \frac{6(1-\nu^2)\cdot P_0}{k^3}\frac{\overline{E}h^3}{E_m h_0^3} \quad (8)$$
$$J = \frac{h_0^3}{12}$$

where γ denotes the bending stiffness parameter, J is the axial moment of inertia of the beam, and $q(x_1)$ denotes the external distributed force.

Equation (8) is obtained in the classical form. The effect of the elastic foundation is incorporated through the bending stiffness parameter γ. By solving (8) via direct integration while applying the appropriate boundary conditions for beams resting on a two-parameter elastic foundation, an analytical solution is obtained, allowing the deflection function $W_0(x_1)$ to be determined.

Based on (3), the beam forces are represented by the internal bending moment and the transverse shear force, expressed as:

$$M = -Jg_1\frac{d^2W_0(x_1)}{dx_1^2},\ Q = Jg_2\frac{d^3W_0(x_1)}{dx_1^3}$$
$$g_1 = 12\int_{-1/2}^{1/2}E(z)\phi_0(z)zdz \quad (9)$$
$$g_2 = 12\int_{-1/2}^{1/2}E(z)\psi_0(z)dz$$

where M denotes the bending moment, Q represents the transverse shear force, and g_1 and g_2 are parameters associated with the internal bending moment and the transverse shear force of the beam.

Taking into account (1)-(3), the parameters of the internal bending moment and the transverse shear force can be written as:

$$g_1 = 12\int_{-1/2}^{1/2}\left(E_m + (E_c - E_m)\left(\frac{1}{2} + \frac{z}{h_0}\right)^p\right)(-C_0\cdot z + z^2)dz$$
$$g_2 = 12\int_{-1/2}^{1/2}\left(E_m + (E_c - E_m)\left(\frac{1}{2} + \frac{z}{h_0}\right)^p\right)\left(A_0 - C_0\cdot z + \frac{z^2}{2}\right)dz \quad (10)$$

where the parameters A_0 and C_0, taking into account relation (7), are determined based on (6).

The analysis of an FG beam resting on the proposed novel two-parameter elastic foundation is carried out in the following steps:

- Determination of the deflection function $W_0(x_1)$ by solving (8).
- Evaluation of the internal beam forces (*M, Q*) using (9) with consideration of (10).
- Determination of the displacement components from (2), taking into account (6) and (7).
- Evaluation of the normal and shear stress components using (3), with consideration of (2), (6), and (7).

III. NUMERICAL RESULTS AND DISCUSSION

The numerical results are expressed in the form of the displacements and stresses of FG beams resting on a two-parameter elastic foundation. The obtained results are employed to evaluate and validate the accuracy of the proposed theoretical formulation. The beam is composed of materials with the following properties: ceramic: alumina ($E_c = 380\,GPa$, $v = 0.3$), metal: aluminum ($E_m = 70\,GPa$, $v = 0.3$).

For convenience, the following non-dimensional parameters are introduced:

Axial displacement: $\overline{U}_1^0 = 100\frac{E_m h_0^3}{q_0 l^4} U_1^0\left(-\frac{l}{2}, \frac{h_0}{2}\right)$

Transverse displacement: $\overline{W}_0 = 100\frac{E_m h_0^3}{q_0 l^4} W_0(0,0)$

Axial stress: $\overline{\sigma}_1^0 = \frac{h}{q_0 l}\sigma_1^0\left(0, -\frac{h_0}{2}\right)$

Transverse shear stress: $\overline{\tau}_{13}^0 = \frac{h}{q_0 l}\tau_{13}^0\ \left(-\frac{l}{2}, 0\right)$

In the considered example, the bending behavior of an FG beam resting on a two-parameter elastic foundation and subjected to a sinusoidal transverse load is examined. Table I summarizes the numerical values of displacements and stresses predicted by the present formulation and compares them with the results obtained using the parabolic beam theory [25], the first-order shear deformation theory [26], the inverse hyperbolic beam theory [14], and the hyperbolic beam theory [17]. The comparisons are carried out for two span-to-thickness ratios, l/h_0=5 and l/h_0=20, and for different values of the power-law index p.

The numerical results obtained in this study provide a detailed assessment of the bending behavior of FG beams resting on a two-parameter elastic foundation. A direct comparison with existing beam theories shows that the displacement and stress responses predicted by the present formulation are in very close agreement with those obtained using higher-order shear deformation models, including the inverse hyperbolic theory in [14], the hyperbolic beam theory in [17], and the parabolic beam theory in [24]. As illustrated in Figure 2, this agreement is particularly evident in the displacement responses, confirming the accuracy of the proposed formulation in capturing the global bending behavior of FG beams supported by elastic foundations.

TABLE I. NON-DIMENSIONAL DISPLACEMENTS AND STRESSES OF AN FG BEAM RESTING ON A NOVEL TWO-PARAMETER ELASTIC FOUNDATION AND SUBJECTED TO A SINUSOIDAL LOAD

p	Theory	l/h_0=5				l/h_0=20			
		$\overline{U_1^0}$	$\overline{W_0}$	$\overline{\sigma_1^0}$	$\overline{\tau_{13}^0}$	$\overline{U_1^0}$	$\overline{W_0}$	$\overline{\sigma_1^0}$	$\overline{\tau_{13}^0}$
0	Present	0.7255	2.5021	3.0925	0.4786	0.1786	2.2842	12.174	0.4801
	[25]	0.7251	2.5020	3.0916	0.4769	0.1784	2.2838	12.171	0.4774
	[26]	0.7129	2.0523	3.0396	0.2653	0.1782	2.2839	12.158	0.2653
	[14]	0.7253	2.5019	3.0922	0.4800	0.1784	2.2839	12.171	0.4806
	[17]	0.7253	2.5019	3.0913	0.4755	0.1784	2.2839	12.171	0.4760
1	Present	1.7798	4.9448	4.7870	0.5255	0.4403	4.5777	18.818	0.5253
	[25]	1.7793	4.9458	4.7857	0.5243	0.4400	4.5773	18.813	0.5249
	[26]	1.7588	4.8807	4.6979	0.5376	0.4397	4.5734	18.792	0.5370
	[14]	1.7796	4.9441	4.7867	0.5248	0.4400	4.5774	18.814	0.5245
	[17]	1.7796	4.9458	4.7851	0.4755	0.4400	4.5774	18.814	0.4760
5	Present	2.8654	7.7743	6.6073	0.5278	0.7072	6.9544	25.799	0.5317
	[25]	2.8644	7.7723	6.6057	0.5314	0.7069	6.9540	25.794	0.5323
	[26]	2.8250	7.5056	6.4382	0.9942	0.7062	6.9373	25.752	0.9942
	[14]	2.8649	7.7739	6.6079	0.5274	0.7069	6.9541	25.795	0.5313
	[17]	2.8656	7.7748	6.6968	0.3980	0.7069	6.9543	25.816	0.3988
10	Present	2.9998	8.6542	7.9105	0.4242	0.7382	7.6424	30.927	0.4268
	[25]	2.9989	8.6530	7.9080	0.4226	0.7379	7.6421	30.999	0.4233
	[26]	2.9488	8.3259	7.7189	1.2320	0.7372	7.6215	30.875	1.2320
	[14]	2.9995	8.6539	7.9102	0.4237	0.7380	7.6422	30.923	0.4263
	[17]	3.0004	8.6522	8.0174	0.4344	0.7380	7.6422	30.948	0.4352
∞	Present	3.9376	13.585	3.0927	0.4812	0.9681	12.335	12.174	0.4816
	[25]	3.9363	13.582	3.0916	0.4769	0.9686	12.398	12.171	0.4774
	[26]	3.8702	12.552	3.0396	0.3183	0.9676	12.398	12.158	0.3183
	[14]	3.9371	13.582	3.0922	0.4800	0.9677	12.329	12.171	0.4806
	[17]	3.9379	13.576	3.1282	0.4877	0.9686	12.398	12.180	0.4884

In contrast, the first-order shear deformation theory yields noticeably lower values of both displacements and stresses, leading to a systematic underestimation of the structural response. This discrepancy becomes more pronounced for thick beams, where transverse shear deformation plays a significant role. The results presented in Figure 2 demonstrate that classical beam theories combined with simplified foundation models are insufficient for accurately representing the coupled effects of shear deformation and foundation stiffness.

The influence of material gradation is further highlighted by the parametric analysis with respect to the power-law index. As shown in Figure 3, beam deflections increase monotonically with increasing values of the power-law index, indicating a reduction in overall stiffness as the material distribution shifts toward a metal-rich configuration. At the same time, the stress distributions converge to identical values in the limiting cases of fully ceramic ($p = 0$) and fully metallic ($p \to \infty$) beams. This behavior confirms that the present formulation correctly reproduces the expected response of homogeneous materials as special cases. The effect of the two-parameter elastic foundation is observed in all response quantities. Compared to beams without foundation support, the presence of the elastic substrate leads to a significant reduction in both displacements and stresses, as evidenced by the results in Figures 2 and 3. This reduction is particularly important for beams with smaller slenderness ratios, where the interaction between transverse shear deformation and foundation stiffness becomes more pronounced.

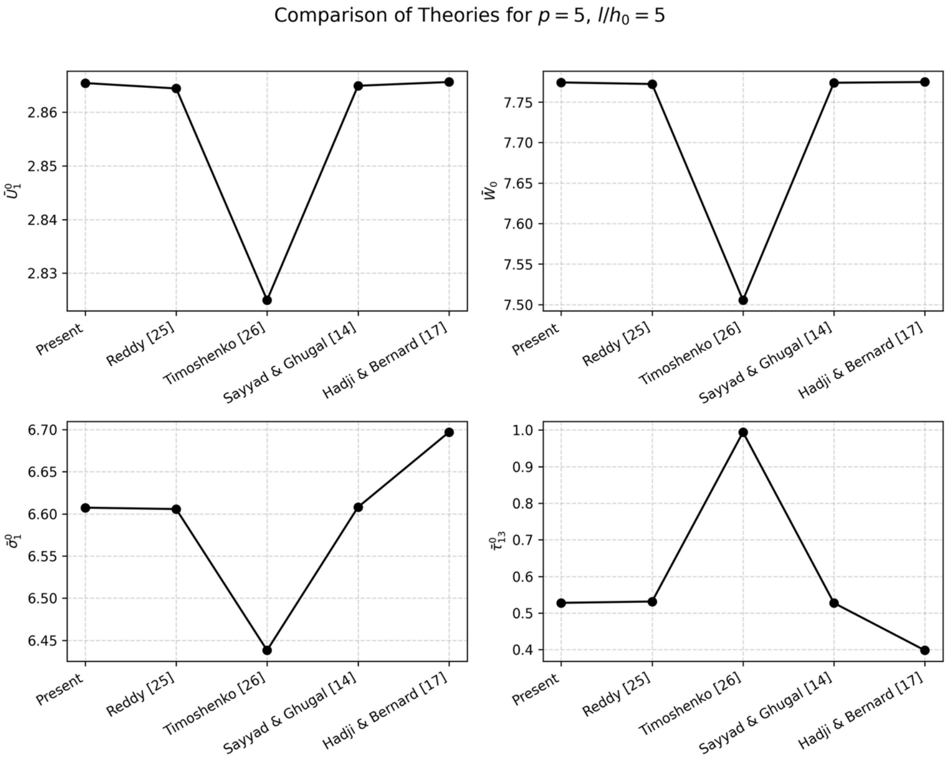

Fig. 2. Comparison of non-dimensional displacements and stresses predicted by different beam theories for p=5 and l/h_0=5.

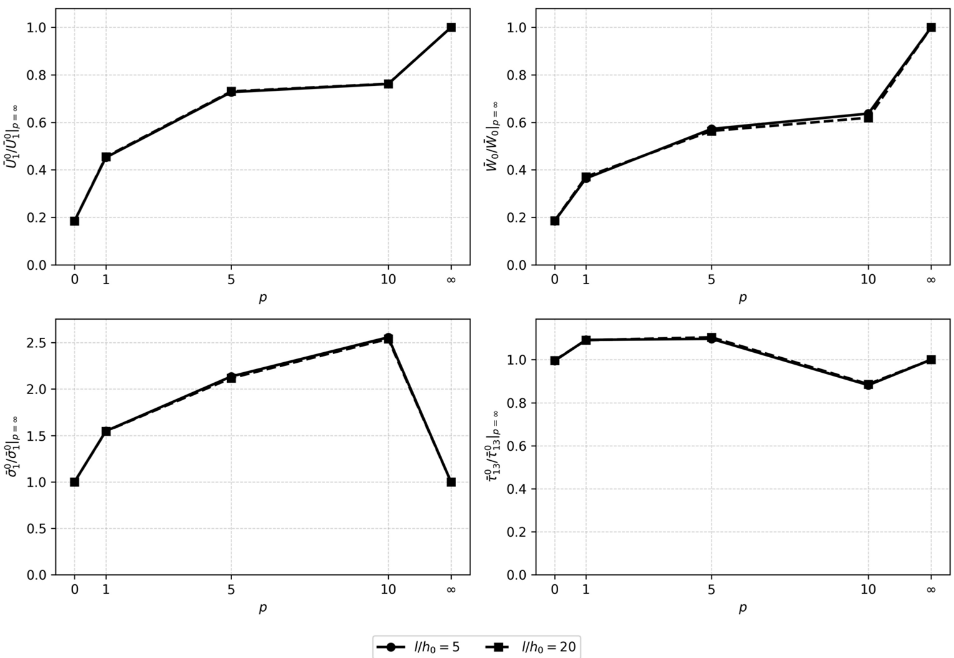

Fig. 3. Asymptotic convergence of normalized displacements and stresses of a FG beam as $p \to \infty$ (metallic limit) for different slenderness ratios.

Additional insights are provided by the asymptotic investigation with respect to the power-law index, as shown in Figure 3. By normalizing the response quantities with respect to their values in the homogeneous metallic limit, it is observed that all mechanical responses smoothly converge as $p \to \infty$, even in the presence of the two-parameter elastic foundation. This result confirms that the introduction of additional foundation parameters does not compromise the mathematical consistency or physical admissibility of the model, while preserving the correct limiting behavior.

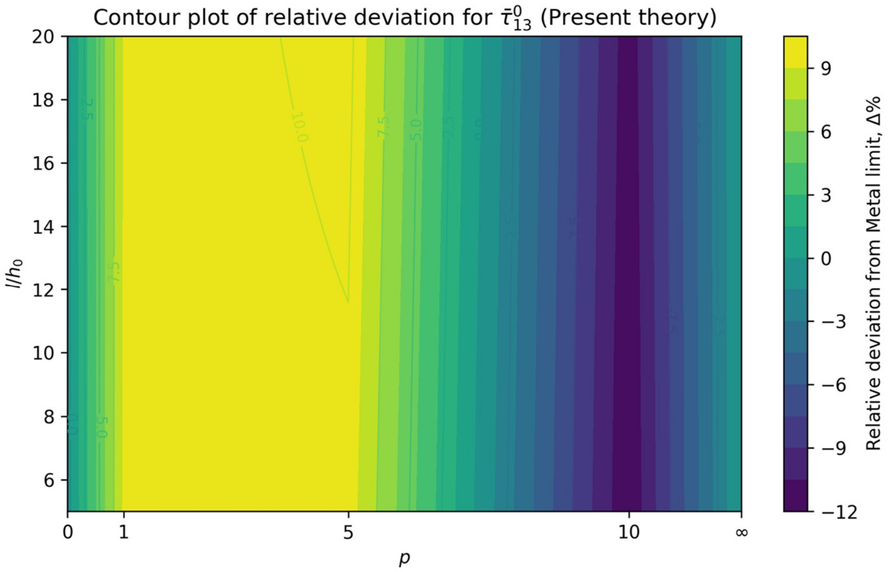

Fig. 4. Contour map of the relative deviation of normalized transverse shear stress from the metallic limit as a function of the power-law index and beam slenderness (present theory).

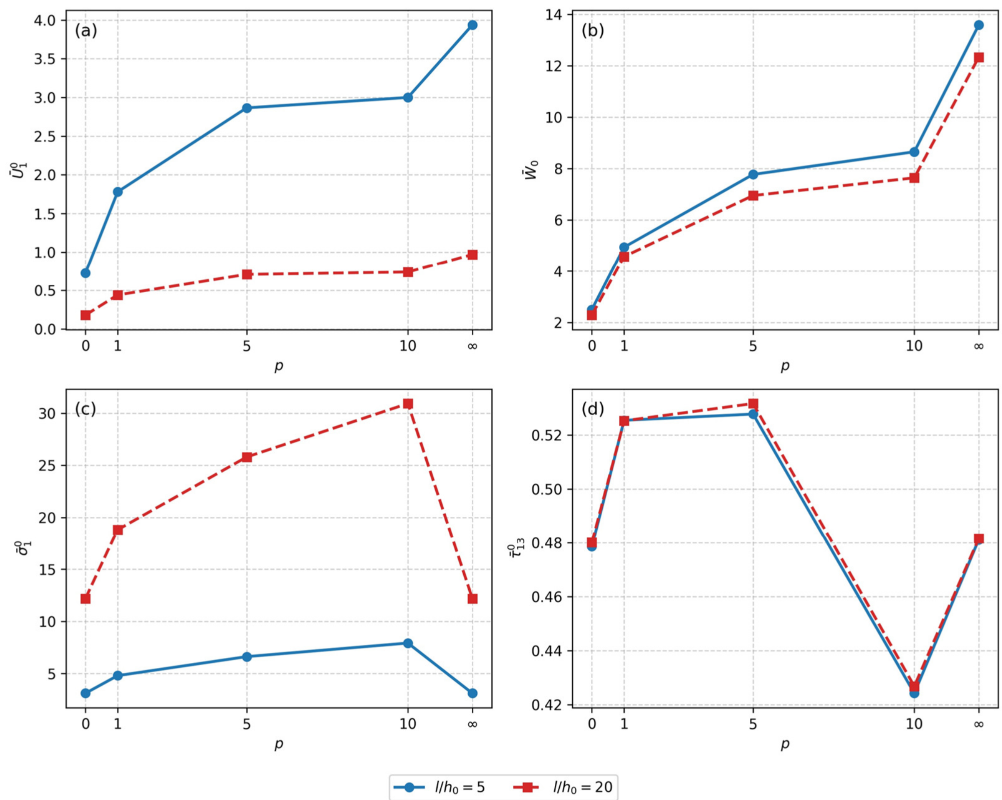

Fig. 5. Effect of the power-law index on normalized displacements and stresses of an FG beam for different slenderness ratios.

The contour plots presented in Figure 4 illustrate the relative deviation of the response quantities from the metallic limit and reveal a strong coupling between material gradation, beam slenderness, and foundation parameters. The largest deviations are observed for low values of the power-law index and for thick beams resting on an elastic foundation, where both transverse shear deformation and foundation reactions significantly influence the stress distribution. As the beam becomes slenderer or as the material gradation approaches the metallic limit, the influence of foundation parameters gradually diminishes.

Finally, the displacement- and stress-related responses shown in Figure 5 demonstrates the sensitivity of different mechanical quantities. While displacement responses exhibit a smooth and monotonic dependence on the material gradation parameter, stress components, particularly transverse shear stresses, are significantly more sensitive to variations in material distribution, beam slenderness, and foundation stiffness. This observation underscores the necessity of employing refined beam theories in combination with advanced elastic foundation models for accurate stress prediction in FG beam-foundation systems.

Overall, the results presented in Figures 2-5 demonstrate that the mechanical behavior of FG beams resting on two-parameter elastic foundations cannot be reliably captured using simplified beam or foundation models. The proposed formulation successfully accounts for the complex interaction between material gradation, geometric characteristics, and foundation effects, providing a physically consistent and analytically robust framework for analyzing and benchmarking FG beam-foundation systems.

IV. CONCLUSIONS

This study presented an analytical formulation for the bending analysis of Functionally Graded (FG) beams resting on a novel two-parameter elastic foundation. The proposed approach accounts for the combined effects of material gradation and elastic foundation interaction, enabling an accurate prediction of both displacement and stress responses.

Comparative studies based on numerical and graphical results demonstrate that the present formulation yields displacement responses in very close agreement with advanced higher-order beam theories. At the same time, it provides a significantly improved prediction of transverse shear stresses compared to classical beam models. These observations highlight the limitations of simplified formulations and emphasize the necessity of refined models for accurately describing FG beam–foundation interaction.

The asymptotic investigation with respect to the power-law index confirms that all response quantities smoothly converge to the homogeneous metallic solution as $p \to \infty$, even in the presence of the two-parameter elastic foundation. This behavior verifies the mathematical consistency and physical admissibility of the proposed formulation and ensures correct limiting behavior.

Further parametric analyses reveal a strong coupling between material gradation, beam slenderness, and foundation parameters. The results indicate that stress-related quantities, particularly transverse shear stresses, exhibit higher sensitivity to these parameters than displacement responses. Overall, the proposed formulation provides a robust and versatile analytical framework for the bending analysis of FG beams on elastic foundations and offers reliable benchmark solutions for future theoretical and numerical studies.

DECLARATION OF COMPETING INTERESTS

The authors declare that they have no competing interests.

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DATA AVAILABILITY

The data supporting the findings of this study are available from the corresponding author upon reasonable request.

AI USE AND DECLARATION OF GENERATIVE AI USE

The authors declare that no generative AI tools were used in the preparation of this manuscript.

REFERENCES

[1] Winkler, *Theory of Elasticity and Strength*. Czechoslovakia: Dominicus Prague, 1867.

[2] P. L. Pasternak, *" On a New Method of Analysis of an Elastic Foundation by Means of Two Constants*, Moscow, Russia: Gosudarstvennoe Izdatelstvo Literaturi po Stroitelstvui Arkhitekture, 1954.

[3] A. D. Kerr, "Elastic and Viscoelastic Foundation Models," *Journal of Applied Mechanics*, vol. 31, no. 3, pp. 491–498, Sep. 1964, https://doi.org/10.1115/1.3629667.

[4] V. Z. Vlasov and U. N. Leont'ev, *Beams, Plates, and Shells on Elastic Foundation*, Washington, D.C., USA: National Aeronautics and Space Administration, Jan. 1966.

[5] T. Nguyen-Van and T. Bui-Tien, "Investigation of the Eigenvector of Stochastic Finite Element Methods of Functionally Graded Beams with Random Elastic Modulus," *Engineering, Technology & Applied Science Research*, vol. 13, no. 4, pp. 11253–11257, Aug. 2023, https://doi.org/10.48084/etasr.5991.

[6] W. Pompe *et al.*, "Functionally Graded Materials for Biomedical Applications," *Materials Science and Engineering: A*, vol. 362, no. 1–2, pp. 40–60, Dec. 2003, https://doi.org/10.1016/S0921-5093(03)00580-X.

[7] D. T. Thuy, L. Nguyen Ngoc, D. N. Tien, and H. V. Thanh, "An Analytical Solution for the Dynamics of a Functionally Graded Plate Resting on Viscoelastic Foundation," *Engineering, Technology & Applied Science Research*, vol. 13, no. 1, pp. 9926–9931, Feb. 2023, https://doi.org/10.48084/etasr.5420.

[8] M. Şimşek, "Fundamental Frequency Analysis of Functionally Graded Beams by Using Different Higher-order Beam Theories," *Nuclear Engineering and Design*, vol. 240, no. 4, pp. 697–705, Apr. 2010, https://doi.org/10.1016/j.nucengdes.2009.12.013.

[9] A. S. Sayyad and Y. M. Ghugal, "Analytical Solutions for Bending, Buckling, and Vibration Analyses of Exponential Functionally Graded Higher Order Beams," *Asian Journal of Civil Engineering*, vol. 19, no. 5, pp. 607–623, Jul. 2018, https://doi.org/10.1007/s42107-018-0046-z.

[10] A. Sayyad and Y. Ghugal, "Bending, Buckling and Free Vibration Responses of Hyperbolic Shear Deformable FGM Beams," *Mechanics of Advanced Composite Structures*, vol. 5, no. 1, Apr. 2018, https://doi.org/10.22075/macs.2018.12214.1117.

[11] T. P. Vo, H.-T. Thai, T.-K. Nguyen, and F. Inam, "Static and Vibration Analysis of Functionally Graded Beams Using Refined Shear Deformation Theory," *Meccanica*, vol. 49, no. 1, pp. 155–168, Jan. 2014, https://doi.org/10.1007/s11012-013-9780-1.

[12] Ş. D. Akbaş, "Free Vibration and Bending of Functionally Graded Beams Resting on Elastic Foundation," *Research on Engineering Structures and Materials*, vol. 1, no. 1, 2015, https://doi.org/10.17515/resm2015.03st0107.

[13] A. Chikh, "Investigations in Static Response and Free Vibration of a Functionally Graded Beam Resting on Elastic Foundations," *Frattura ed Integrità Strutturale*, vol. 14, no. 51, pp. 115–126, Dec. 2019, https://doi.org/10.3221/IGF-ESIS.51.09.

[14] A. S. Sayyad and Y. M. Ghugal, "An Inverse Hyperbolic Theory for FG Beams Resting on Winkler-Pasternak Elastic Foundation," *Advances in Aircraft and Spacecraft Science*, vol. 5, no. 6, pp. 671–689, Nov. 2018, https://doi.org/10.12989/AAS.2018.5.6.671.

[15] L. A. Chaabane *et al.*, "Analytical Study of Bending and Free Vibration Responses of Functionally Graded Beams Resting on Elastic Foundation," *Structural Engineering and Mechanics*, vol. 71, no. 2, pp. 185–196, 2019, https://doi.org/10.12989/SEM.2019.71.2.185.

[16] H. Ait Atmane, A. Tounsi, and F. Bernard, "Effect of Thickness Stretching and Porosity on Mechanical Response of a Functionally Graded Beam Resting on Elastic Foundations," *International Journal of Mechanics and Materials in Design*, vol. 13, no. 1, pp. 71–84, Mar. 2017, https://doi.org/10.1007/s10999-015-9318-x.

[17] L. Hadji and F. Bernard, "Bending and Free Vibration Analysis of Functionally Graded Beams on Elastic Foundations with Analytical Validation," *Advances in Materials Research*, vol. 9, no. 1, pp. 63–98, Mar. 2020, https://doi.org/10.12989/AMR.2020.9.1.063.

[18] Z. Li, Y. Xu, and D. Huang, "Accurate Solution for Functionally Graded Beams with Arbitrarily Varying Thicknesses Resting on a Two-parameter Elastic Foundation," *The Journal of Strain Analysis for Engineering Design*, vol. 55, no. 7–8, pp. 222–236, Oct. 2020, https://doi.org/10.1177/0309324720922739.

[19] L. Hadji, F. Bernard, and N. Zouatnia, "Bending and Free Vibration Analysis of Porous-Functionally-Graded (PFG) Beams Resting on Elastic Foundations," *Fluid Dynamics & Materials Processing*, vol. 19, no. 4, pp. 1043–1054, 2023, https://doi.org/10.32604/fdmp.2022.022327.

[20] M. H. Ghazwani, "New Enriched Beam Element for Static Bending Analysis of Functionally Graded Porous Beams Resting on Elastic Foundations," *Mechanics of Solids*, vol. 58, no. 5, pp. 1878–1893, Oct. 2023, https://doi.org/10.3103/S0025654423600885.

[21] H. Ezzati, A. Rastgoo, and F. Ebrahimi, "Thermal Buckling Analysis of Functionally Graded Graphene Origami-Enabled Auxetic Metamaterial Beams Rested on an Elastic Foundation," *Structures*, vol. 70, Dec. 2024, Art. no. 107764, https://doi.org/10.1016/j.istruc.2024.107764.

[22] J. Chen, W. Qu, C. Ye, Z. Zhao, and H. Wang, "The Effect of Hyperelasticity and Nonlinearity on the Dynamic Behaviors of Hyperelastic Functionally Graded Beams on Nonlinear Elastic Foundation," *International Journal of Structural Stability and Dynamics*, vol. 25, no. 09, May 2025, Art. no. 2550096, https://doi.org/10.1142/S0219455425500968.

[23] S. Akhazhanov, "Simplified Method of Calculating a Beam on a Two-Parameter Elastic Foundation," *International Journal of GEOMATE*, vol. 25, no. 111, Nov. 2023, https://doi.org/10.21660/2023.111.3898.

[24] S. Akhazhanov, D. Baltabai, and B. Nurlanova, "Refined Mechanical and Mathematical Model of an Elastic Half-Plane," *Technobius*, vol. 2, no. 1, Mar. 2022, Art. no. 0014, https://doi.org/10.54355/tbus/2.1.2022.0014.

[25] J. N. Reddy, "A Simple Higher-Order Theory for Laminated Composite Plates," *Journal of Applied Mechanics*, vol. 51, no. 4, pp. 745–752, Dec. 1984, https://doi.org/10.1115/1.3167719.

[26] S. P. Timoshenko, "On the Correction for Shear of the Differential Equation for Transverse Vibrations of Prismatic Bars," *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, vol. 41, no. 245, pp. 744–746, May 1921, https://doi.org/10.1080/14786442108636264.