Stochastic Modeling and Dynamic Optimization for the Long-Term Sustainability of Energy Savings in Demand-Response Systems

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ABSTRACT

The global energy transition critically depends on the effectiveness and long-term persistence of energy-saving measures. However, traditional evaluation models tend to be deterministic, failing to capture the stochastic nature of phenomena such as consumer fatigue and the "rebound effect." This study proposes a rigorous mathematical framework based on stochastic optimal control. We introduce the Dynamic Sustainability Coefficient (DSC), $\psi(t)$, which is modeled using an Ornstein-Uhlenbeck-type Stochastic Differential Equation (SDE). The net savings maximization problem is solved using the Hamilton-Jacobi-Bellman (HJB) equation. The numerical results, based on calibrated synthetic data, show that the adaptive optimal control strategy outperforms static strategies, increasing the project's Net Present Value (NPV) by 20.1%. A clear methodological workflow is provided to facilitate the model's reproducibility and application in energy policy.

Keywords-energy analysis; sustainability; stochastic optimal control; Hamilton-Jacobi-Bellman equation; demand response; mathematical modeling

I. INTRODUCTION

The intersection of energy security, economic stability, and climate change mitigation has made energy efficiency and conservation cornerstones of 21st-century global policy. Beyond the imperative of transitioning to renewable sources, demand management, often called the "first fuel," is the most cost-effective and readily available tool for meeting the objectives of the Paris Agreement [1–3]. In this context, Demand Response (DR) programs, which incentivize consumers to reduce or shift their consumption, have become essential to modern Smart Grids. However, assessing the true impact of these measures is hampered by a persistent methodological assumption: linearity. Cost-benefit analyses frequently quantify savings through simple extrapolations or deterministic decay models [4–6]. This assumption is fundamentally flawed, as energy savings are inherently dynamic and stochastic. The phenomenon of "savings erosion" is multifaceted. Technologically, obsolescence can degrade performance [7–9]. Economically, the "rebound effect" (Jevons Paradox) can partially or fully offset initial savings [3]. Perhaps the most complex factor is human: in DR programs, "response fatigue" is a critical challenge [10–13].

Existing literature has largely addressed these factors in isolation. Bottom-up engineering models often fail to capture the randomness of human behavior [14]. Top-down econometric models lack the granularity to target specific programs [11–13]. Optimization models tend to be deterministic and focus on short time horizons [15–17]. A critical gap thus exists: the lack of a mathematical framework that integrates (1) natural decay, (2) stochastic shocks, and (3) the capacity for intervention (control) to manage the "sustainability" of these savings [18–20]. This article addresses this limitation by presenting a novel contribution: the formalization of the DSC as a controlled stochastic process [21–23]. Unlike previous studies, we integrate SDE theory with dynamic optimization using the HJB equation. This methodology not only quantifies uncertainty but also provides an optimal, closed-form "decision rule" for managing DR programs, filling a critical gap in the energy planning literature [24–27].

II. METHODOLOGY

To ensure clarity and reproducibility of the research design, the study follows a structured quantitative workflow, as illustrated in Figure 1. This methodological framework links the physical conceptualization of energy savings with the mathematical rigor of optimal control. The sequence begins with the definition of stochastic variables (Step 1), proceeds to model the DSC $\psi(t)$ via an SDE (Step 2), and concludes with the derivation of the optimal policy using the HJB equation (Step 3). Finally, the theoretical model is validated using numerical solutions and Monte Carlo simulations (Steps 4–5) to quantify the economic performance of the proposed strategy.

A. The Stochastic Control Model

We consider a planner who aims to maximize the sustainable energy savings from a program over a finite time horizon T.T.

B. The Dynamic Sustainability Coefficient (DSC)

We introduce the state variable ϕ (t) $\in$ [0,1]. The instantaneously observed energy saving, S(t), is given by:

$$S(t) = \psi(t)S_{max} \quad (1)$$

We model the evolution of $\psi(t)$ as an SDE under the real-world probability measure P:

$$d\psi(t) = \left[k\left(\theta - \psi(t)\right) + \gamma u(t)\right]dt + \sigma dW(t) \quad (2)$$

where:

- $\psi(0) = \psi_0$: initial sustainability level.
- $\kappa > 0$: rate of decay (mean-reversion speed).
- $\theta \in [0,1]$: long-term sustainability level (without intervention).
- $u(t) \in [0,u_{max}]$: control variable (intervention effort).
- $\gamma > 0$: effectiveness of the control. $\sigma \geq 0$: volatility (stochastic shocks).
- d$W(t)$: increment of a standard Wiener process.

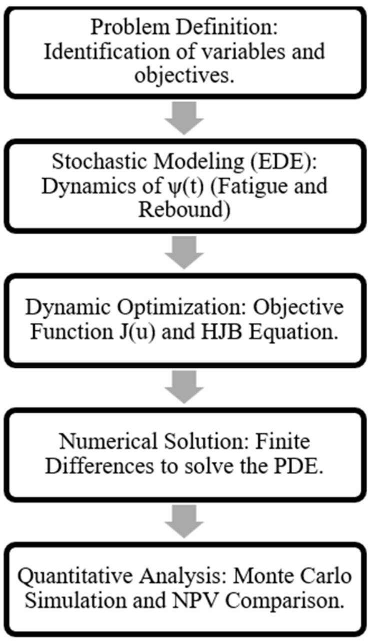

Fig. 1. Workflow from conceptualization to numerical validation.

C. The Planner's Objective Function

The cost of applying control is $c(u(t))$, assumed to be quadratic:

$$c\left(u(t)\right) = \frac{1}{2}\eta u(t)^2 \quad (3)$$

where $\eta > 0$ is the cost parameter. The planner maximizes the expected present value of net savings (savings minus cost), discounted at a rate $\rho > 0$. The objective function J is:

$$J(\psi_0, u) = \mathrm{E}\left[\int_0^T e^{-\rho t}\left(\psi(t)S_{max} - \frac{1}{2}\eta u(t)^2\right)dt + e^{-\rho t}K(\psi(T))\right] \tag{4}$$

where $E[\cdot]$ is the expectation at $t = 0$, and $K(\psi(T))$ is the terminal value (assumed $K = 0$). The problem is to find the optimal strategy $u^*(t)$ that defines the value function $V(\psi,t)$:

$$V(\psi, t) = \sup_{u\in[0,u_{max}]} J(\psi, t; u) \tag{5}$$

D. The Hamilton-Jacobi-Bellman (HJB) Equation

The value function $V(\psi,t)$ must satisfy the HJB equation:

$$\sup_{u\in[0,u_{max}]}\left\{\left(\psi S_{max} - \frac{1}{2}\eta u^2\right) + V_t + V_\psi[\kappa(\theta-\psi) + \gamma u] + \frac{1}{2}\sigma^2 V_{\psi\psi}\right\} - \rho V = 0 \tag{6}$$

where V_t, V_ψ, $V_{\psi\psi}$ are the partial derivatives of V. The first-order condition for u (maximizing the term in the braces) gives:

$$-\eta u + \gamma V_\psi = 0 \Rightarrow u = \frac{\gamma V_\psi}{\eta} \tag{7}$$

Considering the constraints $u \in [0,u_{max}]$, the optimal control is:

$$u^*(\psi, t) = \min\left(\max\left(0, \frac{\gamma V_\psi}{\eta}\right), u_{max}\right) \tag{8}$$

Substituting u* (in its unconstrained form) back into the HJB, we obtain the non-linear PDE for V (ψ,t):

$$(\psi S_{max}) + V_t + V_\psi\kappa(\theta-\psi) + \frac{1}{2}\sigma^2 V_{\psi\psi} + \frac{\gamma^2 V_\psi^2}{2\eta} - \rho V = 0 \tag{9}$$

This PDE is solved numerically backward in time from the terminal condition $V(\psi,T) = 0$.

III. SIMULATION AND RESULTS ANALYSIS

We solve the HJB equation numerically (using a finite-difference method) and then simulate paths (using an Euler-Maruyama scheme).

A. Simulation Configuration and Data

It is important to note that, given the model's theoretical basis and the challenges of isolating stochastic variables in real long-term settings, this study employs synthetic data for numerical validation. The parameters were calibrated based on the empirical literature on demand response programs (see Table I), ensuring the simulation mirrors realistic scenarios with a 20-year horizon and a 0.05 discount rate.

B. Model Parametrization

For the numerical simulation, we use a set of baseline parameters (summarized in Table I) chosen to represent a medium-term demand management program. The social discount rate ($\rho = 5\%$) is a standard value for evaluating public energy projects. The decay rate ($k = 0.20$) and long-term level ($\theta = 0.1$) were chosen to model a "rapid fatigue" scenario, in which program effectiveness without intervention erodes significantly in the first few years [28]. This is consistent with observational studies on the persistence of efficiency savings. The volatility ($\sigma = 0.15$) reflects moderate uncertainty, capturing exogenous factors such as energy price shocks or unexpected weather variations [29]. Finally, the control effectiveness ($\gamma = 0.5$) and cost ($\eta = 10$) are calibration parameters chosen to illustrate a clear trade-off between intervention cost and marginal effectiveness. The sensitivity analysis examines the impact of these values [30].

TABLE I. BASELINE MODEL PARAMETRIZATION

Parameter	Symbol	Value	Description
Horizon	T	20 years	Program lifespan
Max. Savings	S_{max}	100 MWh/year	Theoretical max. savings (normalized)
Discount Rate	ρ	5%	Social interest rate
Decay Rate	κ	0.20	20% annual reversion (rapid fatigue)
Long-term level	θ	0.1	Program decays to 10% effectiveness
Volatility	σ	0.15	15% annual volatility (shocks)
Control effectiveness	γ	0.5	Intervention effectiveness
Control Cost	η	10	Quadratic cost parameter
Initial Sustain	$\psi 0$	1.0	Program starts at perfect effectiveness

C. The Optimal Control Policy u(ψ,t)*

The numerical solution of the HJB is the optimal control policy. As shown in Figure 2, the optimal policy is highly nonlinear and state-dependent ($u*(\psi,t)$). The X-axis represents time ($t = 0$ to $T = 20$), the Y-axis represents sustainability ($\psi = 0$ to 1), and the color represents the intensity of the intervention u^* (Red=High, Blue=Low).

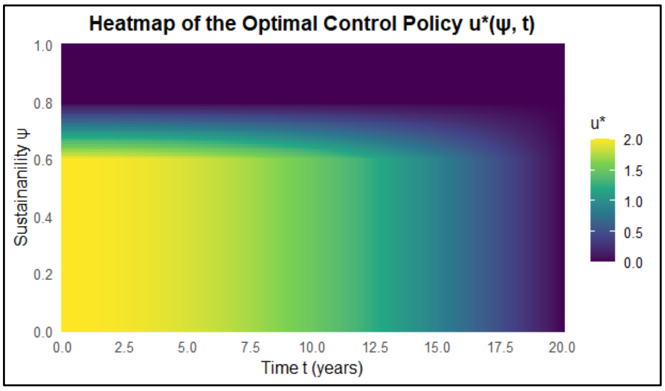

Fig. 2. Heatmap of the optimal control policy.

- Non-Intervention Region (Blue/Cold): When ψ is high (e.g., $\psi > 0.8$), $u^* \approx 0$, making intervention inefficient.
- Active Intervention Region (Red/Hot): When ψ falls below a threshold (e.g., $\psi \approx 0.6$), u^* increases sharply to prevent the system from entering a low-sustainability state.

- Time Effect (X-Gradient): Near the end of the horizon ($t \to T$), the intervention decreases as the value of saving the program for the remaining time diminishes.

D. Trajectory Simulation (Monte Carlo)

We compare three strategies:

- Baseline (No Intervention): $u(t) = 0$
- Constant (Naive) Policy: $u(t) = u$ (fixed budget).
- Optimal (HJB) Policy: $u(t)= u^*(\psi,t)$ (adaptive).

Figure 3 indicates that under the baseline policy ($u = 0$), sustainability $E[\psi(t)]$drops quickly and converges toward the long-run floor $\theta = 0.1$. The Constant policy ($u = \bar{u}$) keeps ψ at a higher level than the baseline, but it remains inefficient and continues to decay over time. By contrast, the Optimal policy (u^*) maintains sustainability at a high, stable level (e.g., $E[\psi] \approx 0.75$), effectively acting as a "thermostat".

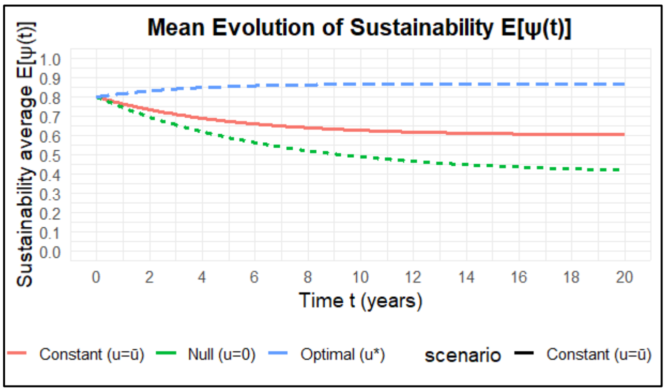

Fig. 3. Mean Evolution of Sustainability E[ψ(t)].

E. Cost-Benefit Analysis

We compare the total expected net present value, $V = J(u)$, across strategies. The total cost of the Optimal and Constant policies is similar (120.3 vs. 115.1). However, the Optimal policy "spends better." It generates a total NPV that is 20.1% higher than the Constant policy and 94.0% higher than doing nothing (Table II and Figure 4).

F. Sensitivity Analysis

To assess the robustness of our findings, we conduct a sensitivity analysis on two critical parameters: process volatility (σ) and intervention cost (η). We compare the "Baseline Case" (Table II) with a "High Volatility" scenario ($\sigma = 0.30$) and a "High Intervention Cost" scenario ($\eta = 20$). The results are summarized in Table II.

The sensitivity analysis confirms the superiority of the adaptive optimal strategy. In the High Volatility scenario (Row 2), the value of all policies decreases due to greater uncertainty. However, the "Optimal Uplift" (the added value from using the HJB policy) increases significantly. This is intuitive: when the future is more uncertain, the value of a flexible strategy that can react to shocks (acting as "insurance") is much greater than that of a fixed policy. In the High-Cost scenario (Row 3), intervention is more expensive. The Baseline Policy is unaffected. The Constant Policy is heavily penalized, as its fixed expenditure is now inefficient. The Optimal Policy adapts by intervening less frequently, and while its total NPV decreases, it still substantially outperforms the Constant Policy. In all cases, the u∗ policy derived from the HJB framework robustly outperforms naive strategies, demonstrating its value under different market conditions.

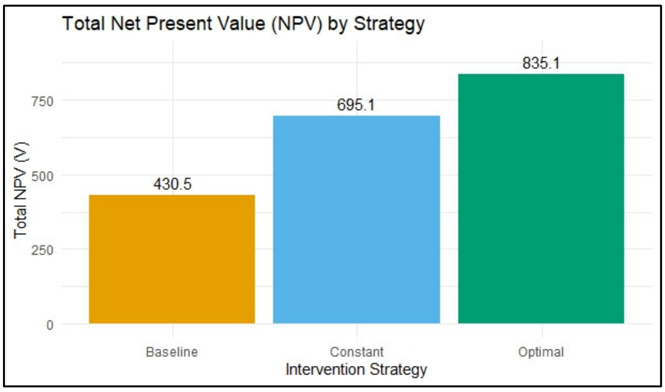

Fig. 4. Total Net Present Value (NPV).

TABLE II. SENSITIVITY ANALYSIS OF TOTAL NPV (V) UNDER DIFFERENT SCENARIOS

Scenario	Baseline Policy	Constant Policy	Optimal Policy	Optimal Uplift
Baseline case ($\sigma = 0.15, \eta = 10$)	430.5	695.1	835.1	+20.1
High volatility ($\sigma = 0.30, \eta = 10$)	410.2	615.5	780.9	+26.9
High cost ($\sigma = 0.15, \eta = 20$)	430.5	580.0	710.3	+22.5

IV. CONCLUSION

This study has developed and validated a stochastic control framework to address the sustainability of energy savings. The main novelty lies in the model's ability to dynamically adjust the intervention's intensity based on the system's current state, rather than following pre-established schedules. Quantitative results show that adaptive management is superior not only in total energy savings but also in economic efficiency. Furthermore, this stochastic control framework offers considerable flexibility. It can be adapted to incorporate volatile energy prices (by treating them as a second stochastic factor) or to manage portfolios of multiple savings measures. The robustness of this approach suggests that “sustainability engineering” can move from an intuition-based art to a rigorous quantitative discipline, providing planners with the tools they need to navigate the uncertainty inherent in the energy transition.

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