Segmented UAV Trajectory Optimization via Fifth-Order Polynomial Approximation and Analytical Optimal Control

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ABSTRACT

This study aims to optimize the flight trajectory of Unmanned Aerial Vehicles (UAVs) with given final conditions. An approach for constructing the optimal UAV trajectory through successive sections using a fifth-degree polynomial approximation is presented. A mathematical model is developed that considers the features of using local inertial coordinate systems on each trajectory section. A motion optimization method is proposed based on minimizing the quadratic functional that characterizes both the accuracy of reaching specified points and the energy cost for control. The solution to the optimization problem is obtained using the analytical design of the optimal controller. The analytical synthesis is novel and yields closed-form expressions for the optimal control on each segment, ensuring minimum-energy trajectories that meet the specified waypoints exactly. The proposed approach is validated through a practical example and MATLAB/Simulink simulation, demonstrating accurate and stable path-following by a UAV.

Keywords-unmanned aerial vehicle; trajectory optimization; polynomial approximation; inertial coordinate systems; quadratic functional; Bolza problem; optimal control; segmented trajectory; analytical design

I. INTRODUCTION

Modern UAVs execute multi-stage missions in dynamic environments, necessitating adaptive trajectory planning that accounts for time, energy, and other external factors. Indeed, hybrid algorithms incorporating machine learning techniques have been shown to improve the operational efficiency of such systems by 10% compared to traditional methods [1].

UAV swarm operations introduce a further layer of complexity, requiring the coordinated planning of multiple UAV trajectories. This encompasses collision avoidance among swarm members, synchronized arrival at target locations, optimal task distribution, and communication link maintenance. Effective swarm mission planning can improve overall system efficiency by a factor of two to three compared to the operation of individual UAVs [2]. As most UAV applications are deployed in low-altitude urban environments, trajectory planning must address numerous constraints. These primarily include inter-UAV safe intervals (vertical and horizontal), hovering altitude, communication coverage area, and time and priority constraints. Furthermore, unpredictable factors inherent to UAV operations, such as wind gusts, system health degradation, and potential collisions with static and dynamic objects, necessitate the development of online decision-making schemes to ensure flight safety [3].

Classical optimization methods are based on the mathematical principles of variational calculus and optimal control theory. The process of trajectory optimization consists of constructing a trajectory that provides an extremum of the selected quality criterion while satisfying the specified constraints, representing a method for deriving programmed control [4]. Numerical heuristic optimization methods have gained wide popularity due to their ability to solve complex problems without the need to differentiate the objective function. Swarm intelligence algorithms, including ant colony optimization and genetic algorithms, demonstrate advantages such as rapid convergence. However, they suffer from issues related to linearity and portability, as evidenced by a comparative analysis of the hybrid Archimedes optimization algorithm and the optimization algorithm [5]. Consequently, the main drawbacks of existing approaches include the high computational complexity of classical methods, the tendency of heuristic algorithms to fall into local minima, the opacity of machine learning methods, and limited adaptability to changing environmental conditions. Therefore, modern research is aimed at creating hybrid methods that combine the mathematical rigor of classical approaches with the adaptability and efficiency of modern artificial intelligence algorithms [6].

Despite significant advancements, several gaps remain in current approaches to trajectory optimization. Although classical variational methods are mathematically rigorous, they struggle to adapt in real-time to dynamic constraints, require heuristic algorithms, demand high computational resources, and risk local minima convergence. Recent bio-inspired methods, such as artificial bee colony and firefly algorithms, lack guarantees for analytical optimality. The present work aims to address these issues by: (1) employing polynomial approximations for trajectory segments to reduce computational complexity while guaranteeing continuity, (2) enabling real-time implementation through analytical optimal control expressions, and (3) balancing energy efficiency and positioning accuracy via the minimization of a quadratic functional. The primary contribution of this work lies in its synthesis of mathematical rigor with computational efficiency, resulting in a practical framework suitable for real-time UAV operations.

II. MATERIALS AND METHODS

This paper addresses the problem of optimal trajectory movement of a UAV. The primary objective is to develop a method for forming an optimal UAV flight trajectory that ensures the passage through specified points in space while minimizing energy costs and satisfying positioning accuracy requirements [7]. Mathematically, the problem is formulated as determining the optimal control of the UAV that ensures movement along a trajectory passing through specified points in space with coordinates $(x^{(k)}, y^{(k)}, z^{(k)})$ in the inertial reference system (*OXYZ*), where k is the point number. It is necessary to minimize the function that characterizes both the accuracy of reaching the specified points and the energy costs of control [8].

The flight is performed along three successive given trajectories: $\gamma_1(t_0, t_1), \gamma_2(t_1, t_2), \gamma_3(t_2, t_3)$, where $\gamma_1(t_0, t_1)$ is the initial section, $\gamma_2(t_1, t_2)$ is the main section of the flight, and $\gamma_3(t_2, t_3)$ is the section that includes the return of the UAV to the starting point. In turn, (t_0, t_3) is the section of the full flight.

Methods for the analytical synthesis and optimization of UAV control systems for flight along successive sections are based on the optimal control theory, taking into account real-world application constraints [9]. The trajectory is described using an appropriate polynomial approximation, which provides an effective solution to control optimization problems:

$$\gamma(t) = \sum_{i=0}^{n} C_i t^i \qquad (1)$$

where $\gamma(t)$ represents the spatial coordinates, t is the UAV flight time ($t_0 \leq t \leq t_3$), and $C_i, i = \underline{1, \mathrm{n}}$ are the coefficients to be determined [10].

The polynomial is used to approximate the projection of the UAV trajectory onto each axis of the inertial reference system:

$$P_5(t) = C_0 + C_1 t + C_2 t^2 + C_3 t^3 + C_4 t^4 + C_5 t^5 \qquad (2)$$

To uniquely define the polynomial, it is necessary to specify six conditions (according to the number of coefficients C_0, C_1, C_2, C_3, C_4, C_5). These conditions may include the position, velocity, and acceleration of the UAV at the initial and final points of the trajectory [11].

A fifth-degree polynomial ensures the continuity of not only the position function itself, but also its first and second derivatives, which corresponds to the continuity of velocity and acceleration. This is important for the feasibility of a real UAV trajectory, considering its dynamic characteristics. A polynomial of degree less than five does not provide sufficient flexibility to simultaneously satisfy the conditions for position, velocity, and acceleration at the boundary points. Polynomials of higher degrees can lead to undesirable trajectory fluctuations between specified points, which complicates control and increases energy costs. A fifth-degree polynomial provides a compromise between the accuracy of approximation and the computational complexity of trajectory planning algorithms [12].

The polynomial (2) is used to approximate the projections of the UAV trajectory onto each of the axes of the inertial reference system. Thus, the spatial trajectory of the UAV is represented by three independent polynomials:

$$x(t) = C_{0x} + C_{1x}t + C_{2x}t^2 + C_{3x}t^3 + C_{4x}t^4 + C_{5x}t^5 \quad (3)$$

$$y(t) = C_{0y} + C_{1y}t + C_{2y}t^2 + C_{3y}t^3 + C_{4y}t^4 + C_{5y}t^5 \quad (4)$$

$$\mathrm{z(t) = C_{0z} + C_{1z}t + C_{2z}t^2 + C_{3z}t^3 + C_{4z}t^4 + C_{5z}t^5} \quad (5)$$

where $x(t), y(t), z(t)$ are the coordinates in an inertial frame of reference at a given time t.

To determine the polynomial coefficients, it is necessary to set the boundary conditions for each trajectory section. These conditions include:

1. The coordinates of the UAV at the start and final points of the section:

$$x(t_0) = x_0, y(t_0) = y_0, z(t_0) = z_0 \quad (6)$$

2. Velocity projections at the initial and final points:

$$\dot{x}(t_0) = v_{x0},\ \ \dot{x}(t_f) = v_{xf}, \dot{y}(t_0) = v_{y0},\ \ \dot{y}(t_f) = v_{yf}, \dot{z}(t_0) = v_{z0},\ \ \dot{z}(t_f) = v_{zf} \quad (7)$$

3. Acceleration projections at start and end points (if required):

$$\ddot{x}(t_0) = a_{x0},\ \ \ddot{x}(t_f) = a_{xf}, \ddot{y}(t_0) = a_{y0},\ \ \ddot{y}(t_f) = a_{yf}, \ddot{z}(t_0) = a_{z0},\ \ \ddot{z}(t_f) = a_{zf} \quad (8)$$

The optimization framework involves the following practical UAV constraints:

- Dynamic limits: Maximum lateral acceleration $|\mathrm{a}_z| \leq \mathrm{a}_{max}$ (typically 4-6 m/s^2) and roll/pitch angles ±30°.
- Environmental factors: The current model assumes nominal conditions; extendable to wind disturbances: $\dot{X}^{(k)} = V_x^{(k)} + W_x(t)$.
- Optimization objective: Functional (26) balances terminal accuracy (c_1,c_2 terms ensure waypoint tracking) with energy minimization (c_3 term reduces control effort and mechanical stress).

The Analytical Design of the Optimal Controller (ACOC) provides immediate control solutions compared to iterative methods (e.g., genetic algorithms, particle swarm), which require 500-2000 iterations and take s to min to compute. This analytical approach ensures optimality for quadratic functionals and onboard implementation feasibility. While predictive model control offers adaptability through per-timestep optimization, the proposed segmented approach computes coefficients in advance, updating only in response to major deviations.

Defining all these conditions allows for the unique calculation of the polynomial coefficients for each coordinate [13]. To determine the kinematic characteristics of the UAV's motion, it is necessary to find the derivatives of the polynomial function (2).

The UAV's motion is modeled within an inertial reference system, where the principle of inertia is preserved. Therefore, if the UAV is not subject to external forces, it will move at a constant speed [14]. In practice, the axes of such a system are defined by three pairwise orthogonal lines (*X*, *Y*, *Z*) used to specify the UAV's position and describe its trajectory.

Differentiating the polynomial from (2) twice yields:

$$\dot{P}_5(t) = C_1 + 2C_2t + 3C_3t^2 + 4C_4t^3 + 5C_5t^4 \quad (9)$$

$$\ddot{P}_5(t) = 2C_2 + 6C_3t + 12C_4t^2 + 20C_5t^3 \quad (10)$$

Equations (9) and (10) define the projections of velocity and acceleration. The coefficients $(C_0, \ldots, C_5)$ are determined by solving the system of calculations for the polynomials $\left(P_5(t), \dot{P}_5(t), \ddot{P}_5(t)\right)$ of the end of guidance [15]. These expressions are used to calculate the projections of the UAV's velocity and acceleration onto the axes of the inertial coordinate system:

$$v_{x(t)} = C_{1x} + 2C_{2x}t + 3C_{3x}t^2 + 4C_{4x}t^3 + 5C_{5x}t^4 \quad (11)$$

$$v_{y(t)} = C_{1y} + 2C_{2y}t + 3C_{3y}t^2 + 4C_{4y}t^3 + 5C_{5y}t^4 \quad (12)$$

$$v_{z(t)} = C_{1z} + 2C_{2z}t + 3C_{3z}t^2 + 4C_{4z}t^3 + 5C_{5z}t^4 \quad (13)$$

$$a_{x(t)} = 2C_{2x} + 6C_{3x}t + 12C_{4x}t^2 + 20C_{5x}t^3 \quad (14)$$

$$a_{y(t)} = 2C_{2y} + 6C_{3y}t + 12C_{4y}t^2 + 20C_{5y}t^3 \quad (15)$$

$$a_{z(t)} = 2C_{2z} + 6C_{3z}t + 12C_{4z}t^2 + 20C_{5z}t^3 \quad (16)$$

The coefficient values C_1, C_2, C_3, C_4, C_5 for each coordinate axis are determined by solving a system of equations derived from the boundary conditions. This process yields a trajectory that passes through the specified points in space while satisfying the required velocity and acceleration constraints at these points. A typical application problem is the flight of a UAV along a trajectory passing through specified points in space with specified coordinates $\left(x^{(k)}, y^{(k)}, z^{(k)}\right)$ in the inertial reference system (*OXYZ*), where k indicates the point number. The problem of determining an optimal flight trajectory composed of consecutive segments is then considered. For each segment, the guidance law must be determined to meet the requirements of the Control System (CS). The specified trajectory must: (1) pass near the specified points within a given accuracy, and (2) minimize the integral losses associated with the UAV maneuvers and changes in control load. To achieve this, optimality criteria must be formulated, including characteristics for the accuracy of reaching the final position and for the integral losses over the entire control period.

This problem is solved by the classical Boltz problem for minimizing the functional of the form:

$$J = g(x_k, t_k) + \int_{t_0}^{t_k} F(x, t)dt \quad (17)$$

where $g(x_k, t_k)$ describes the final goal of the state $x(t)$ over the section $[t_0, t_k]$, and $\int_{t_0}^{t_k} F(X, t)dt$ determines the integral losses in controlling the UAV motion.

The function $g(x_k, t_k)$ defines the final goal for the UAV flight section between the specified points, including both the initial and final points, as well as internal trajectory points [16].

Consequently, $g(x_k, t_k)$ characterizes the minimum deviation of the UAV flight trajectory from the points: $x_k, k = \underline{1, N}$, where N is the number of intermediate specified points in the trajectory. To determine the UAV's trajectory, a problem similar to the problems of rigid body kinematics is considered, determining the motion of the center of mass. Thus, the functional (5) is transformed into a quadratic functional:

$$J = X_k^T R X_k + \int_{t_0}^{t_k} [X(t)^T Q(t) X(t) + U(t)^T S(t) U(t)] dt \quad (18)$$

where $X_k = [\Delta x_k, \Delta y_k, \Delta z_k]^T$ is the vector of minimal deviations relative to the intermediate route point, $X(t) = [x(t), y(t), z(t)]^T$ is the coordinate vector of the UAV center of mass, and $U(t) = \left[a_x(t), a_y(t), a_z(t)\right]^T$ is the control vector of normal accelerations of the center of mass.

To solve the optimization problem in (18), it is necessary to select the matrices $R, Q(t), S(t)$. The functional (18) includes terms with variables of different dimensions and values: X_k, $X(t)$, $U(t)$. Therefore, these variables must be transformed into dimensionless form through normalization, using the matrices R, $Q(t)$, $S(t)$ with the permissible ranges of the variables serving as the normalizing coefficients. In general, the change in the UAV's motion parameters in space is determined by the vector differential equation:

$$\dot{x}(t) = f(x, t), \quad x(t_0) = x_0 \quad (19)$$

where $f(x, t)$ is the nonlinear vector function. The problem is simplified if the equation of motion (19) takes the following form:

$$\dot{x}(t) = A(t)x(t) + B(t)U(t), \quad x(t_0) = x_0 \quad (20)$$

where $A(t), B(t)$ are coefficient matrices. This problem uses the ACOC problem, in which the desired optimal control vector is:

$$U(t) = -S^{-1}(t) B^T K(t) X(t) \quad (21)$$

where $K(t)$ is the matrix of coefficients determined by the solution of a vector differential equation:

$$\dot{K}(t) = -A^T(t)K(t) - K(t)A(t) + \\ +K(t)B(t)S^{-1}(t)B^T(t)K(t) - Q(t), K(t_k) = R \quad (22)$$

To determine the optimal control *U(t)*, it is necessary to solve the boundary value problem by solving the system of differential equations (19) and (21) together. The problem can be solved using the sweep method. In this method, approximate boundary values are initially specified for one of the systems (19) or (21). Then, systems (19) and (21) are integrated repeatedly in forward and backward time until the specified accuracy is achieved. For the following problem, the kinematics of the UAV guidance scheme toward a given point $\left(O^{(k)}, X^{(k)}, Y^{(k)}, Z^{(k)}\right)$ of the inertial reference system is shown in Figure 1, where *k* is the ordinal number of the point in space along the UAV's path.

A new inertial coordinate system $\left(O^{(k)}, X^{(k)}, Y^{(k)}, Z^{(k)}\right)$ is defined. At each stage of the UAV flight, the origin of the coordinate system $O^{(k)}$ coincides with the trajectory's starting point [17]. The axis $O^{(k)}X^{(k)}$ indicates the direction of movement to the next point. The $O^{(k)}Y^{(k)}$ axis is directed vertically upwards. The combination of the axes $O^{(k)}Y^{(k)}$ and $O^{(k)}X^{(k)}$forms a right-handed coordinate system $O^{(k)}Z^{(k)}$.

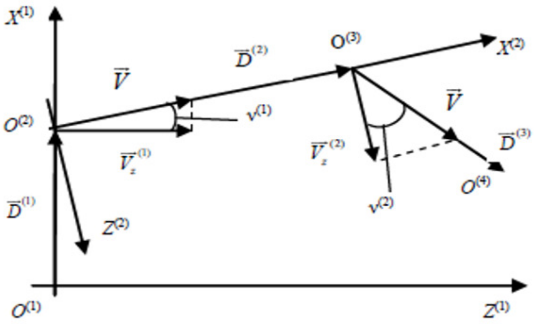

Fig. 1. UAV guidance.

The following notation is introduced:

- $\underline{V}$ denotes the UAV velocity vector. To simplify the mathematical model, its magnitude is assumed constant ($|\underline{V}| = const$).
- $\nu^{(k)}$ is the orientation angle of the UAV velocity vector at the terminus of the *k*-th guidance segment.
- $\underline{D}^{(k)}$is the origin of the given inertial coordinate system for the *k*-th guidance segment.
- $O^{(k)}$– is the origin of the given inertial coordinate system for the *k*-th guidance segment.

The motion of the UAV in the horizontal plane relative to the given inertial coordinate system on the *k*–th guidance segment is described by the following system of linear differential equations:

$$\dot{X}^{(k)} = X_x^{(k)}, \quad X^{(k)}(0) = X_0^{(k)}, \dot{Z}^{(k)} = V_z^{(k)}, \ Z^{(k)}(0) = Z_0^{(k)}, \ \dot{V}_x^{(k)} = a_x^{(k)} \quad (23)$$

$$V^{(k)}(0) = V_{x0}^{(k)}, \dot{V}_z^{(k)} = a_z^{(k)}, \quad V_z^{(k)}(0) = V_{z0}^{(k)} \quad (24)$$

where $X^{(k)}, Z^{(k)}$ are the coordinates of the UAV in the *k*-th coordinate system, $V_x^{(k)}, V_z^{(k)}$ are the projections of the UAV velocity vector $\underline{V}$ on the axes of the *k*-th coordinate system, and $a_x^{(k)}, a_z^{(k)}$ are the accelerations of the UAV in the *k*-th coordinate system.

The lateral acceleration of the UAV, $a_z^{(k)}(t)$, is taken as the control variable. Given that $|\underline{V}| = const$, expression (24) yields:

$$V_x^{(k)} = \sqrt{V^2 - V_x^{(k)2}}, V = |\underline{V}| \quad (25)$$

The optimal control (acceleration) for the UAV on the *k*-th guidance section is determined. A standard optimization criterion, common in quadratic guidance problems, is employed:

$$J = \frac{1}{2}\left[c_1 V_z - V_{given}^2 + c_2 Z - Z_{given}^2\right]_{t=t_j} + \frac{1}{2}\int_{t_0}^{t} c_3 a_z^2 dt \quad (26)$$

where t_j is the time of the UAV's encounter with the required point in space, V_{given} is the specified value of the UAV's velocity projection onto the $O^{(k)}Z^{(k)}$ axis of the corresponding inertial coordinate system at the end of guidance for the *k*-th segment, Z_{given} is the lateral coordinate of the specified point of the trajectory, and c_1, c_2, c_3 are the coefficients. The optimal lateral acceleration $a_z(t)$ that minimizes functional (26) can be determined using variational calculus. Applicable methods include the Pontryagin maximum principle (for finding optimal trajectories), the Lagrange multiplier method (for incorporating system constraints), and the classical variational approach via the Euler-Lagrange equations (for minimizing the control quality functional):

$$a_z(V_z, Z, t) =$$

$$-\Lambda_v(t)\left[V_z(t) - V_{given}\right] - \Lambda_z(t)\left[Z(t) - Z_{given}\right] \quad (27)$$

$$\Lambda_v(t) = \frac{\frac{1}{c_2} + \frac{1}{c_1}(t_f - t)^2 + \frac{1}{3}(t_f - t)^3}{D(t_f - t)} \quad (28)$$

$$\Lambda_z(t) = \frac{\frac{1}{c_1}(t_f - t) + \frac{1}{2}(t_f - t)^2}{D(t_f - t)} \quad (29)$$

$$D(t_f - t) = \left[\frac{1}{c_2} + \frac{1}{3}(t_f - t)^3\right] \cdot \left[\frac{1}{c_1} + (t_f - t)\right] - \frac{1}{4}(t_f - t)^4 \quad (30)$$

In a particular case, $c_1 \to \infty$ can also be taken as $c_2 \to \infty$. This corresponds to neglecting the integral term in the functional (18). Expressions (19)-(22) remain valid for the spatial UAV guidance problem. In this case, the UAV's coordinates, velocities, and accelerations are replaced by the corresponding three-dimensional vectors. In these expressions, $t_f - t = t_{remain}$ is the time remaining until the UAV meets the next given point in space [18].

$$V_{given} = V_{given}^{(k)} = V_z^{(k)}(t_f) sin v_{given}^{(k)} \quad (31)$$

where $V_z^{(k)}(t_f)$ is the value of the UAV's velocity at the final time of the *k*-th guidance interval and $v_{given}^{(k)}$ is the specified angle of the UAV to the corresponding point of the trajectory $O^{(k+1)}$. This defines the orientation of the vector *V* relative to the coordinate system $\left(O^{(k)}X^{(k)}Z^{(k)}\right)$.

$$t_{remain} = t_{remain}^{(k)} = \frac{D^{(k)}}{|\dot{D}|^{(k)}} \quad (32)$$

where $D^{(k)}$ is the current range of the UAV to a given point $O^{(k+1)}$ at the *k*-th guidance section and $\left|\dot{D}\right|^{(k)}$ is the closing speed of the UAV towards $O^{(k+1)}$.

In the mathematical model of UAV motion in the horizontal plane $\left(O^{(k)}X^{(k)}Z^{(k)}\right)$, where altitude is not considered, the main variables are:

- $x(t)$ and $z(t)$: the UAV coordinates in the inertial system.
- $v_x(t)$ and $v_z(t)$: the velocity components.
- $a_x(t)$and $a_z(t)$: the acceleration components.
- $\varphi(t)$: the UAV course (angle of inclination of the velocity vector relative to the (*OX*) axis.

The equations of motion for the UAV are therefore:

$$\dot{x}(t) = v_x(t), \quad \dot{z}(t) = v_z(t) \quad (33)$$

$$\dot{v}(t) = a_x(t), \quad \dot{v}(t) = a_z(t) \quad (34)$$

For practical calculations, the approximation $\pi \approx 3.142$ and a step size $h = 0.05$ are used. The resulting numerical values (the initial 12 values) are provided in Tables I and II.

TABLE I. INITIAL 6 VALUES

T	$x(t)$	$y(t)$
0.05	0.145014	0.203644
0.1	0.321394	0.383022
0.15	0.519543	0.540902
0.2	0.733052	0.680173
0.25	0.957556	0.803485
0.3	1.19003	0.913142

TABLE II. INITIAL 6 VALUES

T	$x(t)$	$y(t)$
0.35	1.428345	1.011104
0.4	1.670976	1.099018
0.45	1.916819	1.178264
0.5	2.165064	1.25
0.55	2.415109	1.315198
0.6	2.666506	1.37468

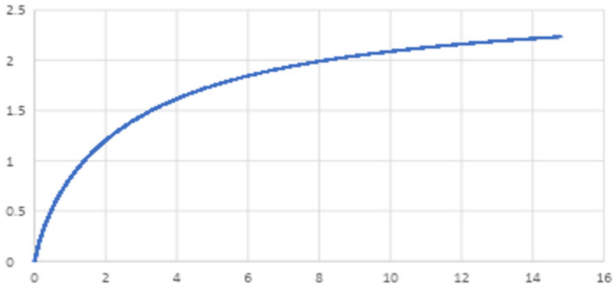

Fig. 2. UAV motion at a constant speed and changing angle with a certain step.

In this case, time is considered $t \in [0{,}2; 4{,}4]$ with a step size $h = 0.2$. The first 12 values are presented in Figure 2. The resulting UAV motion, plotted in different projections $((x(t); z(t))$, is shown in Figure 3. Figure 3 displays the graph of the UAV motion in the (*OZX*) plane. For implementation with constant angular acceleration, it is necessary to enter the coordinates $(z(t); x(t))$ into the UAV motion control point. This is demonstrated using the time $t \in [0{,}2; 4{,}4]$ and the step $h = 0.2$. The quadcopter's position in space is defined by six parameters: three angles and three coordinates of the center of mass. The quadcopter trajectory will be controlled by four parameters: the three coordinates of the quadcopter position and the angle of rotation relative to the vertical axis (yaw angle). The pitch and roll angles affect the horizontal acceleration of the control object and will be calculated according to the specified trajectory, $r_t(t)$:

$$[r_t(t), \psi_t(t)] = [x_t(t), y_t(t), z_t(t), \psi_t(t)] \quad (35)$$

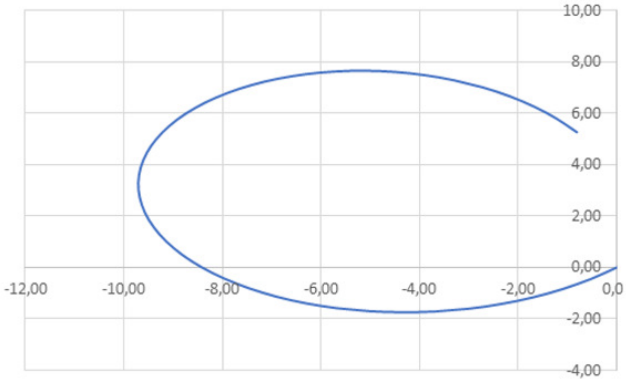

Fig. 3. UAV motion with constant speed and angular acceleration on the plane (*OZ*).

III. RESULTS AND DISCUSSION

Figure 4 presents the quadcopter UAV model developed in MATLAB/Simulink, in which:

- The *X* axis starts at the center of gravity and is directed along the nose of the quadcopter.
- The *Y* axis starts at the center of gravity and is directed to the right of the quadcopter.
- The *Z* axis starts at the center of gravity and is directed down from the quadcopter.

To validate the proposed approach, the UAV model and controller were implemented in MATLAB/Simulink using the UAV Toolbox [9]. The quadcopter's body-fixed axes (Figure 4) are defined for force and moment control, and the overall system is presented as a Simulink block diagram in Figure 5. The internal control structure and the physical frame of the quadcopter model are portrayed in Figure 7. All simulations were conducted with zero initial conditions within a virtual 3D campus environment (Figure 6), where mission waypoints were placed. In this environment, the planned UAV path is highlighted in red, illustrating the computed optimal trajectory. The simulated UAV (marker/blue line) tracks this path with minimal deviation. The results confirm the efficacy of the control algorithm, demonstrating stable and accurate trajectory tracking for a single quadcopter.

Fig. 4. Coordinate and force CS on a quadcopter.

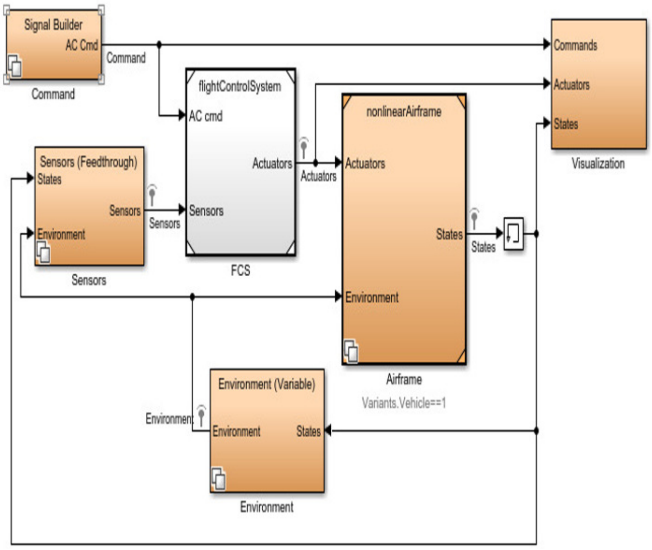

Fig. 5. Simulink block diagram used for simulation.

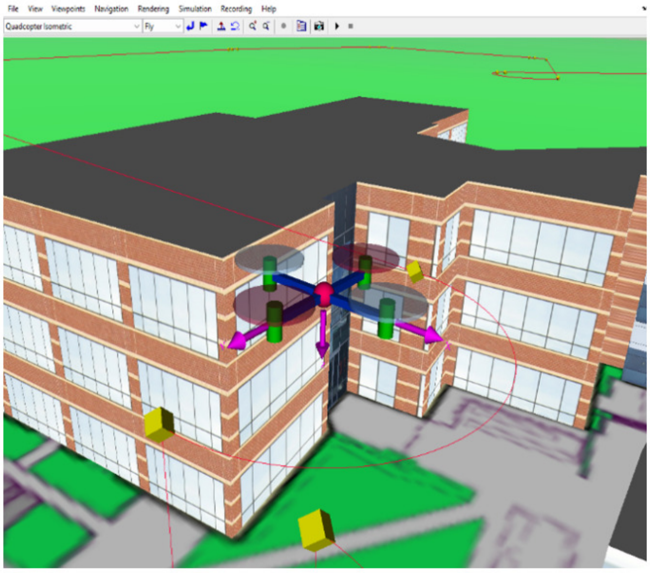

Fig. 6. UVA trajectory over the 3D campus model.

The multi-waypoint trajectory was successfully executed by the quadcopter. Turbulent or stormy conditions would require adaptive replanning extensions. C^2 continuity (position, velocity, acceleration) at waypoints is ensured by the segment approach via proper initialization. The cascaded control loops and physical frame with rotor dynamics are shown in Figure 7, implementing (23-27). The real-time computation without iterative optimization is achieved by the analytical optimal control solution (27-30).

Mission-specific tuning is achieved through the optimization functional (26) via weighting coefficients c_1, c_2, c_3. Terminal accuracy is prioritized by higher values of c_1 and c_2, while higher c_3 focuses more on energy efficiency. Adaptation to different mission requirements is enabled by this flexibility.

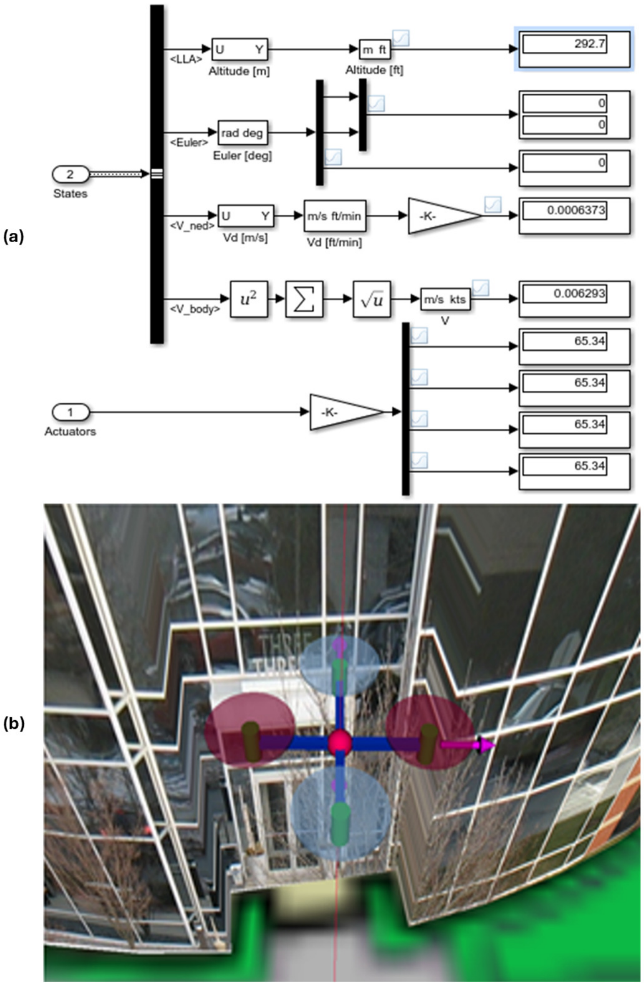

Fig. 7. Simulation model in MATLAB/Simulink: (a) internal structure, (b) real 3D model of the frame.

The key novelty of this work is the integration of segmented polynomial path planning with analytical optimal control design. Unlike standard UAV mission-planning tools, the proposed method jointly computes both the path and the optimal control inputs. By approximating each trajectory segment with a fifth-degree polynomial in a local inertial frame [1], the current study ensures continuity of position, velocity, and acceleration across waypoints, and derives a closed-form optimal control law by minimizing a quadratic cost of tracking error and energy. This contrasts with common planners and autopilot software, which generally apply numerical or reactive controllers without such optimality guarantees. The result is a minimum-energy flight path that exactly meets the specified conditions. In practical terms, the proposed analytical solution greatly reduces online computation and avoids trial-and-error tuning, while providing a guaranteed precision in waypoint tracking [11]. The main contributions are, thus, the segment-wise polynomial trajectory model and its analytic optimal controller, enabling efficient and precise UAV path planning beyond what off-the-shelf planning software typically provides.

IV. CONCLUSIONS

The current study presented an analytical method for UAV flight-path optimization under given waypoint constraints. The trajectory is constructed segment-by-segment using fifth-order polynomials in local frames, and an optimal-control problem is solved on each segment to minimize a quadratic cost of tracking error and control effort [1, 11]. This approach ensures smooth, continuous motion (position, velocity, acceleration) through all waypoints while using minimal control energy. The simulation results confirm stable and accurate path following under the derived controller. The main contributions are the combined polynomial path model and its closed-form optimal controller, which together enable precise waypoint tracking with low computational complexity [11].

Future work will, therefore, involve experimental flight tests, extensions to cooperative multi-UAV missions, and incorporation of real-world uncertainties (e.g., wind disturbances) to further improve robustness and applicability. Applicability is limited by the current assumptions (ideal sensors, nominal atmosphere) in adverse conditions. Specific modifications for the future include: (1) multi-UAV coordination with collision avoidance, (2) adaptive replanning under environmental disturbances, (3) trajectory deformation for obstacle avoidance, and (4) validation with real sensor noise and atmospheric effects. Overall, this work provides a foundation for independent UAV operations that balances theoretical optimality with practical implementation constraints.

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